# PERIYAR UNIVERSITY

NAAC 'A++' Grade - State University
NIRF Rank 56 – State Public University Rank 25
SALEM - 636 011, Tamil Nadu, India.

## CENTRE FOR DISTANCE AND ONLINE EDUCATION
## (CDOE)

## MASTER SCIENCE IN MATHEMATICS
## SEMESTER - II

## ELECTIVE COURSE: MATHEMATICAL MODELLING

**(Candidates admitted from 2024 onwards)**

# PERIYAR UNIVERSITY

## CENTRE FOR DISTANCE AND ONLINE EDUCATION (CDOE)

## M.Sc., MATHEMATICS 2024 admission onwards

**ELECTIVE**
**Mathematical Modelling**

Prepared by:

Centre for Distance and Online Education (CDOE)

Periyar University

Salem - 636011

# Contents

# SYLLABUS: MATHEMATICAL MODELLING

## Objectives:

This course aims to

- Provide rigorous instruction in fundamental mathematical concepts and skills presented in the context of real-world applications.
- Gain a working knowledge of core techniques behind mathematical modelling and develop a basic ability to quantify certain phenomena associated with the physical sciences Represent real-world systems in a mathematical framework.

**Unit I:** Mathematical Modelling through Ordinary Differential Equations of First order Linear Growth and Decay Models - Non-Linear Growth and Decay Models - Compartment Models - Dynamics problems - Simple problems.

**Unit II:** Mathematical Modelling through Systems of Ordinary Differential Equations of First Order Population Dynamics - Epidemics - Compartment Models - Economics - Medicine, Arms Race, Battles and International Trade - Simple problems.

**Unit III:** Mathematical Modelling through Ordinary Differential Equations of Second Order Planetary Motions - Circular Motion and Motion of Satellites - Mathematical Modelling through Linear Differential Equations of Second Order - Miscellaneous Mathematical Models - Simple problems.

**Unit IV:** Mathematical Modelling through Difference Equations Simple Models - Basic Theory of Linear Difference Equations with Constant Coefficients - Economics and Finance - Population Dynamics and Genetics - Probability Theory - Simple problems.

**Unit V:** Mathematical Modelling through Graphs Solutions that can be Modelled through Graphs - Mathematical Modelling in Terms of Directed Graphs, Signed Graphs, Weighted Digraphs - Simple problems.

## References:


1. J.N. Kapur, Mathematical Modelling, Wiley Eastern Limited, New Delhi, 4th Reprint, May 1994.


## Suggested Reading:


1. M. Braun, C.S. Coleman and D. A. Drew, Differential Equation Models, 1994.
2. A.C. Fowler, Mathematical Models in Applied Sciences, Cambridge University Press, 1997.
3. Walter J. Meyer, Concepts of Mathematical Modeling. Courier Corporation, 2012.
4. Edward A. Bender, Introduction to Mathematical Modelling, Dover Publications, 1st ed., 2000.

# UNIT - 1

# Unit 1

# Mathematical Modelling Through First Order Ordinary Differential Equations

## Objectives:

- To model and analyze population growth model (linear & nonlinear) and decay models that change over time.
- To study the spread of technological innovations and infectious diseases
- To discuss the basics of the law of mass action: chemical reactions
- Understand the dynamical problems like simple harmonic motion, motion under gravity in a resisting medium, motion of a rocket and orthogonal trajectory.

## 1.1 Introduction

Mathematical Modelling in terms of differential equations arises when the situation modelled involves some continuous variable(s) varying with respect to some other continuous variable(s) and we have some reasonable hypotheses about the rates of change of dependent variable(s) with respect to independent variable(s).

When we have one dependent variable $x$ (say population size) depending on one independent variable (say time $t$ ), we get a mathematical model in terms of an ordinary differential equation of the first order, if the hypothesis is about the rate of change $dx/dt$.

The model will be in terms of an ordinary differential equation of the second order if the hypothesis involves the rate of change of $dx/dt$.

If there are a number of dependent continuous variables and only one independent variable, the hypothesis may give a mathematical model in terms of a system of first or higher order ordinary differential equations.

If there is one dependent continuous variable (say velocity of fluid $u$ ) and a number of independent continuous variables (say space coordinates $x, y, z$ and time $t$ ), we get a mathematical model in terms of a partial differential equation. If there are a number of dependent continuous variables and a number of independent continuous variables, we can get a mathematical model in terms of systems of partial differential equations.

Mathematical models in term of ordinary differential equations will be studied in this and the next two chapters.

## 1.2 Linear Growth and Decay Models

### 1.2.1 Populational Growth Models

Let $x(t)$ be the population size at time $t$ and let $b$ and $d$ be the birth and death rates, i.e. the number of individuals born or dying per individual per unit time.

Then in time interval $(t, t+\Delta t)$, the numbers of births and deaths would be $bx\Delta t + o(\Delta t)$ and $dx\Delta t + o(\Delta t)$ where $\Delta t$ is an infinitesimal which approaches zero as $\Delta t$ approaches zero, so that

$$x(t+\Delta t) - x(t) = (bx(t) - dx(t))\Delta t + 0(\Delta t), \tag{1.2.1}$$

so that dividing by $\Delta t$ and proceeding to the limit as $\Delta t \to 0$, we get

$$\begin{aligned} \frac{dx}{dt} &= (b-d)x \\ &= ax \quad \text{(say)} \end{aligned} \tag{1.2.2}$$

$$\frac{dx}{x} = adt$$

Integrating (1.2.2), we get

$$\begin{aligned}
\int \frac{dx}{x} &= a\int dt \\
\log x &= at + \log c \\
\log x - \log c &= at \\
\log\left(\frac{x}{c}\right) &= at \\
\frac{x}{c} &= e^{at} \\
x(t) &= c\ exp(at) \quad (\text{since } exp(at) = e^{at})
\end{aligned}$$

By taking $t = 0$, we get

$$x(0) = c\ exp(0)$$

$$c = x(0) \quad (\text{since } e^0{=}1)$$

Then, we have

$$x(t) = x(0)\exp(at)$$

so that the population grows exponentially if $a > 0$, decays exponentially if $a < 0$ and remains constant if $a = 0$ (Figure 1.1)

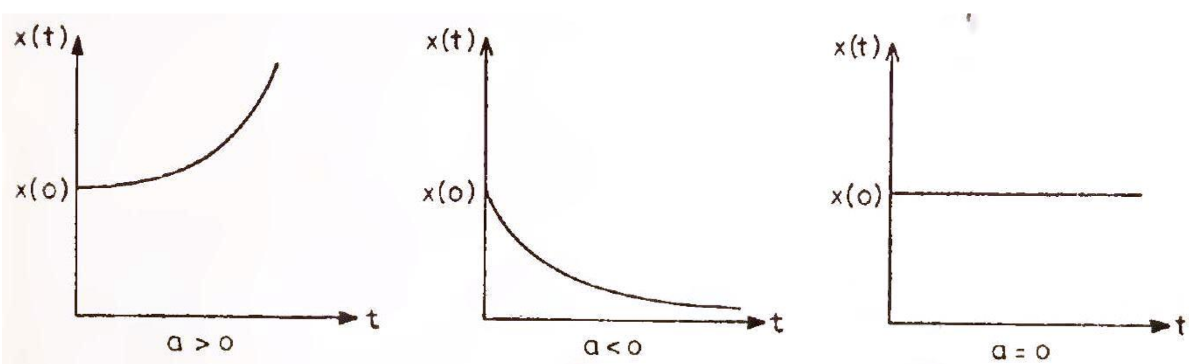



Figure 1.1

(i) If $a > 0$, the population will become double its present size at time $T$, where

$$2x(0) = x(0)\exp(aT)$$

$$\exp(aT) = 2$$

$$T = \frac{1}{a}\ln 2 = (0.69314118)a^{-1}$$

$T$ is called the doubling period of the population and it may be noted that this doubling period is independent of $x(0)$.

It depends only on $a$ and is such that greater the value of $a$ (i.e. greater the difference between birth and death rates), the smaller is the doubling period.

(ii) If $a < 0$, the population will become half its present size in time $T'$ when

$$\begin{aligned} \frac{1}{2}x(0) &= x(0)\exp{(aT')} \\ \exp{(aT')} &= \frac{1}{2} \\ T' &= \frac{1}{a}\ln\frac{1}{2} = -(0.69314118)a^{-1} \end{aligned}$$

It may be noted that $T'$ is also independent of $x(0)$ and since $a < 0, T' > 0$. $T'$ may be called the half-life (period) of the population and it decreases as the excess of death rate over birth rate increases.

### 1.2.2 Growth of Science and Scientists

Let $S(t)$ denote the number of scientists at time $t$, $bS(t)\Delta t + o(\Delta t)$ be the number of new scientists trained in time interval $(t, t + \Delta t)$.

Let $d\Delta S(t)\Delta t + o(\Delta t)$ be the number of scientists who retire from science in the same period, then the above model applies and the number of scientists should grow exponentially.

The same model applies to the growth of Science, Mathematics and Technology.

Thus if $M(t)$ is the amount of Mathematics at time $t$, then the rete of growth of Mathematics is proportional to the amount of Mathematics, so that

$$\frac{dM}{dt} = aM \tag{1.2.3}$$

$$\frac{dM}{M} = adt$$

Integrating (1.2.3), we get

$$\begin{aligned} \int\frac{dM}{M} &= a\int dt \\ \log M &= at + \log c \end{aligned}$$

$$\begin{aligned}
\log M - \log c &= at \\
\log\left(\frac{M}{c}\right) &= at \\
\frac{M}{c} &= e^{at} \\
M(t) &= c\ exp(at)
\end{aligned}$$

By taking $t = 0$, we get

$$M(0) = c\ exp(0)$$

$$c = M(0) \quad (\text{since } e^0{=}1)$$

Then we have

$$M(t) = M(0)\exp(at)$$

Thus according to this model, Mathematics, Science and Technology grow at an exponential rate and double themselves in a certain period of time.

During the last two centuries this doubling period has been about ten years. This implies that if in 1900, we had one unit of Mathematics, then in 1910, 1920, 1930, . . . 1980 we have $2, 4, 8, 16, 32, 64, 128, 256$ unit of Mathematics and in 2000AD we shall have about 1000 units of Mathematics.

This implies that 99.9% of Mathematics that would exist at the end of the present century would have been created in this century and 99.9% of all mathematicians who ever lived, have lived in this century.

The doubling period of mathematics is 10 years and the doubling period of the human population is $30 - 35$ years. These doubling periods cannot obviously be maintained indefinitely because then at some point of time, we shall have more mathematicians than human beings.

Ultimately the doubling period of both will be the same, but hopefully this is a long way away. □

**Remark 1.2.1.** *This model also shows that the doubling period can be shortened by having more intensive training programmes for mathematicians and scientists and by creating conditions in which they continue to do creative work for longer durations in life.*

### 1.2.3 Effects of Immigration and Emigration on Population Size

If there is immigration into the population from outside at a rate proportional to the population size, the effect is equivalent to increasing the birth rate.

Similarly if there is emigration from the population at a rate proportional to the population size, the effect is the same as that of increase in the death rate.

If however immigration and emigration take place at constant rate $i$ and $e$ respectively, equation (1.2.2) is modified to

$$\frac{dx}{dt} = bx - dx + i - e = ax + k \tag{1.2.4}$$

where $a = b - d$, and $k = i - e$.

$$\frac{dx}{dt} = a(x + \frac{k}{a})$$
$$\frac{dx}{x + \frac{k}{a}} = adt$$

Integrating (1.2.4) we get

$$\begin{aligned}
\int \frac{dx}{x + \frac{k}{a}} &= a \int dt \\
\log(x + \frac{k}{a}) &= at + \log c \\
\log\left(x + \frac{k}{a}\right) - \log c &= at \\
\log(\frac{(x + \frac{k}{a})}{c} &= at \\
\frac{(x + \frac{k}{a})}{c} &= e^{at} \\
x(t) + \frac{k}{a} &= c\ exp(at)
\end{aligned}$$

By taking $t = 0$, we get

$$x(0) + \frac{k}{a} = c\ exp(0)$$
$$c = x(0) + \frac{k}{a} \quad \text{(since } e^0\text{=1)}$$

Then we have

$$x(t) + \frac{k}{a} = \left(x(0) + \frac{k}{a}\right) e^{at}$$

The model also applies to growth of populations of bacteria and microorganisms, to the increase of volume of timber in forest, to the growth of malignant cells etc. In the case of forests planting of new plants will correspond to immigration and cutting of trees will correspond to emigration. □

### 1.2.4 Interest Compounded Continuously

Let the amount at time $t$ be $x(t)$ and let interest at rate $r$ per unit amount per unit time be compounded continuously then

$$x(t + \Delta t) = x(t) + rx(t)\Delta t + o(\Delta t)$$

giving

$$\frac{dx}{dt} = xr \tag{1.2.5}$$

$$\frac{dx}{x} = rdt$$

Integrating (1.2.2), we get

$$\begin{aligned}
\int \frac{dx}{x} &= a \int dt \\
\log x &= rt + \log c \\
\log x - \log c &= rt \\
\log(\frac{x}{c}) &= rt \\
\frac{x}{c} &= e^{rt} \\
x(t) &= c\ exp(rt)
\end{aligned}$$

By taking $t = 0$, we get

$$x(0) = c\ exp(0)$$

$$c = x(0) \quad \text{(since } e^0\text{=1)}$$

Then, we have

$$x(t) = x(0)e^{rt} \tag{1.2.6}$$

This formula can also be derived from the formula for compound interest

$$x(t) = x(0)\left(1 + \frac{r}{n}\right)^{nt} \tag{1.2.7}$$

when interest is payable $n$ times per unit time, by taking the limit as $n \to \infty$. In fact comparison of (1.2.6) and (1.2.7) gives us two definitions of the trancendental number $e$ viz.

(i) $e$ is the amount of an initial capital of one unit invested for one unit of time when the interest at unit rate is compounded continuously

(ii) $e = \text{Lt}_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$

Also from (1.2.6) if $x(t) = 1$, then

$$1 = x(0)e^{rt} \implies x(0) = e^{-rt}$$

so that $e^{-rt}$ is the present value of a unit amount due one period hence when interest at the rate $r$ per unit amount per unit time is compounded continuously. □

**Problem 1.2.2.** *List out the decay models and write short notes.*

**Solution**.

**1. Radio-Active Decay**

Many substances undergo radio-active decay at a rate proportional to the amount of the radioactive substance present at any time and each of them has a half-life period. For uranium 238 it is 4.55 billion years.

For potassium it is 1.3 billion years. For theorem it is 13.9 billion years. For rubidium it is 50 billion years while for carbon 14 it is only 5568 years and for white lead it is only 22 years.

In radiogeology, these results are used for radioactive dating. Thus the ratio of radio-carbon to ordinary carbon (carbon 12) in dead plants and animals enables us to estimate their time of death. Radioactive dating has also been used to estimate the age of the solar system and of earth as 45 billion years.

**2. Decrease of Temperature**

According to Newton's law of cooling, the rate of change of temperature of a body is proportional to the difference between the temperature $T$ of the body and temperature $T_s$ of the surrounding medium, so that

$$\frac{dT}{dt} = k\left(T - T_s\right), k < 0$$

and

$$T(t) - T_s = \left(T(0) - T_s\right) e^{kt}$$

and the excess of the temperature of the body over that of the surrounding medium decays exponentially.

**3. Diffusion**

According to Fick's law of diffusion, the time rate of movement of a solute across a thin membrane is proportional of the area of the membrane and to the difference in concentrations of the solute on the two sides of the membrane.

If the area of the membrane is constant and the concentration of solute on one side is kept fixed at $a$ and the concentration of the solution on the other side initially is $c_0 < a$, then Fick's law gives

$$\frac{dc}{dt} = k(a - c), \quad c(0) = c_0$$

so that

$$a - c(t) = (a - c(0))e^{-kt}$$

and $c(t) \to a$ as $t \to \infty$, whatever be the value of $c_0$.

**4. Change of Price of a Commodity**

Let $p(t)$ be the price of a commodity at time $t$, then its rate of change is proportional to the difference between the demand $d(t)$ and the supply $s(t)$ of the commodity in the market so that

$$\frac{dp}{dt} = k(d(t) - s(t))$$

where $k > 0$, since if demand is more than the supply, the price increases. If $d(t)$ and $s(t)$ are assumed linear functions of $p(t)$, i.e. if

$$d(t) = d_1 + d_2 p(t), \quad s(t) = s_1 + s_2 p(t), \quad d_2 < 0, s_2 > 0$$

we get

$$\frac{dp}{dt} = k\left(d_1 - s_1 + (d_2 - s_2)\, p(t)\right) = k(a - \beta p(t)), \quad \beta > 0$$

where $\alpha = d_1 - s_1$, and $\beta = d_2 - s_2$,

or

$$\frac{dp}{dt} = K\left(p_e - p(t)\right).$$

where $p_e$ is the equilibrium price, so that

$$p_e - p(t) = \left(p_e - p(0)\right) e^{-kt}$$

and

$$p(t) \to p_e \quad \text{as} \quad t \to \infty$$

□

# Let us sum up:

- The population growth model.
- Development of science and scientists.
- Changes in population size due to immigration and emigration.
- Interest at the rate $r$ per unit time is compounded continuously.

## Check your progress:

1. What happens to the population size if the difference between birth and death rates are zero?

2. Explain the immigration and emigration to increase of volume of timber in forest?

# 1.3 Non-Linear Growth and Decay Models

### 1.3.1 Logistic Law of Population Growth

As population increases, due to overcrowding and limitations of resources, the birth rate $b$ decreases and the death rate $d$ increases with the population size $x$. The simplest assumption is to take

$$b = b_1 - b_2 x, \;\; d = d_1 + d_2 x, \quad b_1, b_2, d_1, d_2 > 0$$

so that $\frac{dx}{dt} = (b-d)x$, becomes

$$\frac{dx}{dt} = ((b_1 - d_1) - (b_2 - d_2)\,x) = x(a - bx), a > 0, b > 0 \tag{1.3.8}$$

$$\begin{aligned} \frac{dx}{x(a-bx)} &= dt \\ \frac{1}{a}\left(\frac{1}{x} + \frac{b}{a-bx}\right)dx &= dt \end{aligned} \tag{1.3.9}$$

Integrating (1.3.9), we get

$$\begin{aligned} \int \left(\frac{1}{x} + \frac{b}{a-bx}\right)dx &= a\int dt \\ \log x - \log(a-bx) &= at + \log c \\ \log\frac{x}{a-bx} - \log c &= at \\ \log\frac{x}{c(a-bx)} &= at \\ \frac{x}{c(a-bx)} &= e^{at} \end{aligned}$$

$$x(t) = c(a - bx(t))\ exp(at)$$

By taking $t = 0$, we get

$$x(0) = c(a - bx(0))\ exp(0)$$
$$c = \frac{x(0)}{a - bx(0)}$$

Then, we have

$$\frac{x(t)}{a - bx(t)} = \frac{x(0)}{a - bx(0)} e^{at} \quad (1.3.10)$$

Equations (1.3.8) and (1.3.10) show that

(i) $x(0) < a/b \Rightarrow x(t) < a \mid b \Rightarrow dx/dt > 0 \Rightarrow x(t)$ is a monotonic increasing function of $t$ which approaches $a/b$ as $t \to \infty$.

(ii) $x(0) > a/b \Rightarrow x(t) > a/b \Rightarrow dx/dt < 0 \Rightarrow x(t)$ is a monotonic decreasing function of $t$ which approaches $a/b$ as $t \to \infty$.

Now from (1.3.8)

$$\frac{d^2x}{dt^2} = a - 2bx$$

so that $d^2x/dt^2 \gtreqless 0$ according as $x \gtreqless a/2b$. Thus in case (i) the growth curve is convex if $x < a/2b$ and is concave if $x > a/2b$ and it has a point of inflexion at $x = a/2b$. Thus the graph of $x(t)$ against $t$ is as given in Figure 1.2.

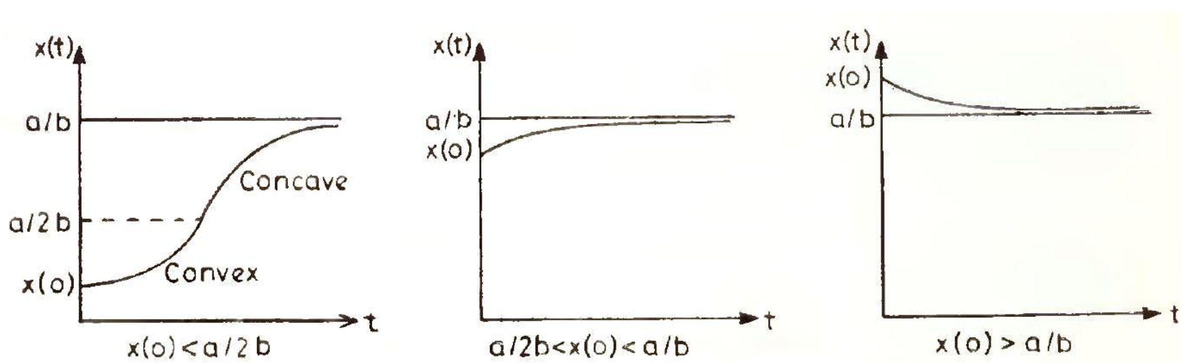



Figure 1.2

If $x(0) < a/2b, x(t)$ increases at an increasing rate till $x(t)$ reaches $a/2b$ and then it increases at a decreasing rate and approaches $a/b$ at $t \to \infty$

If $a/2b < x(0) < a/b, x(t)$ increases at a decreasing rate and approaches $a/b$ as $t \to \infty$

If $x(0) = a/b, x(t)$ is always equal to $a/b$

If $x(0) > a/b, x(t)$ decreases at a decreasing absolute rate and approaches $a/b$ as $t \to \infty$. □

### 1.3.2 Spread of Technological Innovations and Infectious Diseases

Let $N(t)$ be the number of companies which have adopted a technological innovation till time $t$, then the rate of change of the number of these companies depends both on the number of companies which have adopted this innovation and on the number of those which have not yet adopted it, so that if $R$ is the total number of companies in the region

$$\frac{dN}{dt} = kN(R - N) \tag{1.3.11}$$

which is the logistic law and shows that ultimately all companies will adopt this innovation.

Similarly if $N(t)$ is the number of infected persons, the rate at which the number of infected persons increases depends on the product of the numbers of infected and susceptible persons.

As such we again get (1.3.11), where $R$ is the total number of persons in the system.

It may be noted that in both the examples, while $N(t)$ is essentially an integer-valued variable, we have treated it as a continuous variable. This can be regarded as an idealisation of the situation or as an approximation to reality. □

### 1.3.3 Rate of Dissolution

Let $x(t)$ be the amount of undissolved solute in a solvent at time $t$ and let $c_0$ be the maximum concentration or saturation concentration.

i.e. the maximum amount of the solute that can be dissolved in a unit volume of the solvent. Let $V$ be the volume of the solvent.

It is found that the rate at which the solute is dissolved is proportional to the amount of undissolved solute and to the difference between the concentration of the solute at time $t$ and the maximum possible concentration, so that we get

$$\frac{dx}{dt} = kx(t)\left(\frac{x(0)-x(t)}{V} - c_0\right) = \frac{kx(t)}{V}\left((x_0 - c_0V) - x(t)\right)$$

□

### 1.3.4 Law of Mass Action: Chemical Reactions

Two chemical substances combine in the ratio $a:b$ to form a third substance $Z$.

If $z(t)$ is the amount of the third substance at time $t$, then a proportion $az(t)/(a+b)$ of it consists of the first substance and a proportion $bz(t)/\ (a+b)$ of it consists of the second substance.

The rate of formation of the third substance is proportional to the product of the amount of the two component substances which have not yet combined together.

If $A$ and $B$ are the initial amounts of the two substances, then we get

$$\frac{dz}{dt} = k\left(A - \frac{az}{a+b}\right)\left(B - \frac{bz}{a+b}\right)$$

This is the non-linear differential equation for a second order reaction.

Similarly for an $n$th order reaction, we get the non-linear equation

$$\frac{dz}{dt} = k\left(A_1 - a_1z\right)\left(A_2 - a_2z\right)\ldots\left(A_n - a_nz\right)$$

where $a_1 + a_2 + \ldots + a_n = 1$. □

## Let us sum up:

- Logistic law of population growth.
- Transmission of infectious diseases and technological advancements.
- Law of mass action: The third substance is proportional to the product of two component substances.

## Check your progress:

1. Explain logistic growth model
2. Explain the solution of the chemical reactions in Law of Mass Action.

## 1.4 Compartment Models

In the last two chapters, we got mathematical models in terms of ordinary differential equations of the first order, in all of which variables were separable. In the present chapter, we get models in terms of linear differential equations of first order.

We also use here the principle of continuity i.e. that the gain in amount of a substance in a medium in any time is equal to the excess of the amount that has entered the medium in the time over the amount that has left the medium in this time.

### 1.4.1 A Simple Compartment Model

Let a vessel contain a volume $V$ of a solution with concentration $c(t)$ of a substance at time $t$ (Figure 1.3).

Let a solution with constant concentration $C$ in an overhead tank enter the vessel at a constant rate $R$ and after mixing thoroughly with the solution in the vessel, let the mixture with concentration $c(t)$ leave the vessel at the same rate $R$ so that the volume of the solution in the vessel remains $V$.

Using the principle of continuity, we get

$$V(c(t+\Delta t) - c(t)) = Rc\Delta t - Rc(t)\Delta t + 0(\Delta t)$$

giving

$$V\frac{dc}{dt} + Rc = RC$$

$$\frac{dc}{dt} + \frac{R}{V}c = \frac{RC}{V} \tag{1.4.12}$$

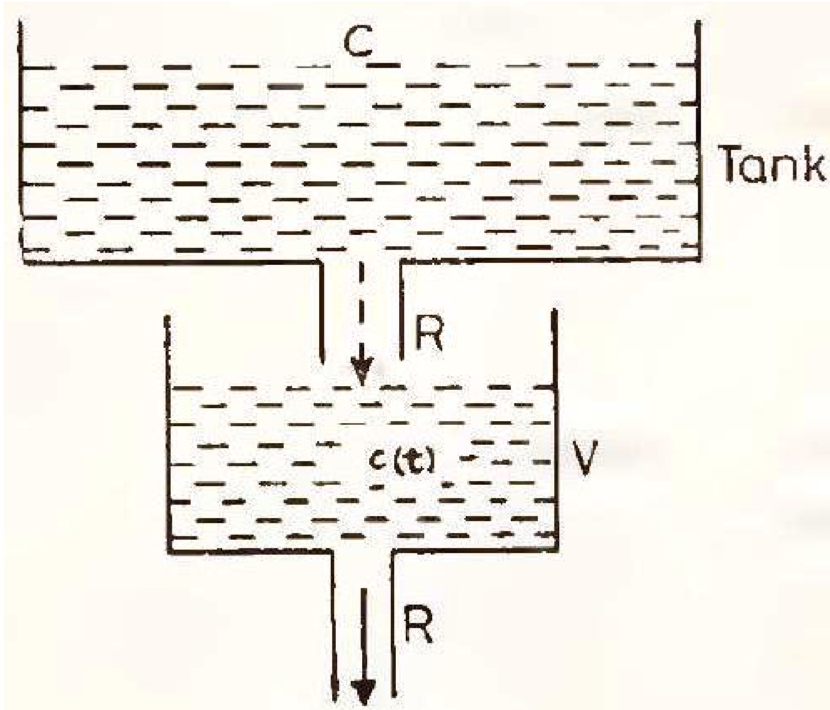



Figure 1.3

Solving (1.4.12), we get

$$\begin{aligned} \text{A. E. is } m + \frac{R}{V} &= 0 \\ m &= -\frac{R}{V} \\ \text{C.F} &= Ae^{-\frac{R}{V}t} \\ P.I &= -\frac{RCe^{0t}}{V(D+\frac{R}{V})} \\ &= Ce^{0t} \\ &= C \\ \therefore\ c(t) &= Ae^{-\frac{R}{V}t} + C \end{aligned}$$

By taking $t = 0$, we get

$$c(0) = Ae^{-\frac{R}{V}0} + C$$

$$A = c(0) - C$$

Then, we have

$$c(t) = (c(0) - C)\exp\left(-\frac{R}{V}t\right) + C$$
$$c(t) = c(0)\exp\left(-\frac{R}{V}t\right) + C\left(1 - \exp\left(-\frac{R}{V}t\right)\right).$$

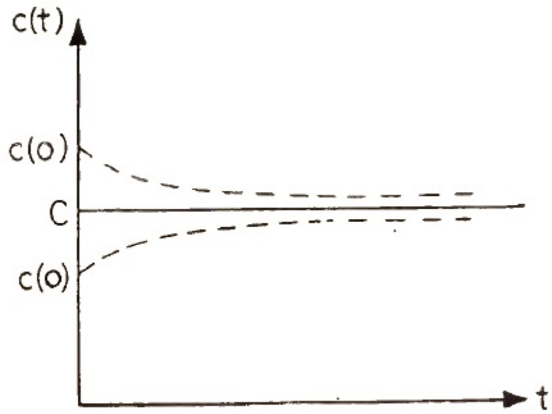



Figure 1.4

As $t \to \infty, c(t) \to C$, so that ultimately the vessel has the same concentration as the overhead tank. Since

$$c(t) = C - (C - c_0) \exp\left(-\frac{R}{V}t\right)$$

If $C > c_0$, the concentration in the vessel increases to $C$, on the other hand if $C < c_0$, the concentration in the vessel decreases to $C$ (see Figure 1.4).

If the rate $R'$ at which the solution leaves the vessel is less than $R$, the equations of continuity gives

$$\begin{aligned} &\frac{d}{dt}\left[(V_0 + (R - R')\,t)\,c(t)\right] \\ &= RC - R'(ct) \end{aligned}$$

where $V$ is the initial volume of the solution in the vessel. This is also a linear differential equation of the first order. □

### 1.4.1 Diffusion of Glucose or a Medicine in the Blood Stream

Let the volume of blood in the human body be $V$ and let the initial concentration of glucose in the blood stream be $c(0)$.

Let glucose be introduced in the blood stream at a constant rate $I$.

Glucose is also removed from the blood stream due to the physiological needs of the human body at a rate proportional to $c(t)$, so that the continuity principle gives

$$V\frac{dc}{dt} = I - kc$$

Now let a dose $D$ of a medicine be given to a patient at regular intervals of duration $T$ each.

The medicine also disappears from the system at a rate proportional to $c(t)$, the concentration of the medicine in the blood stream, then the differential equation given by the continuity principle is

$$V\frac{dc}{dt} = -kc \tag{1.4.13}$$

$$\frac{dc}{c} = \frac{-k}{V}dt$$

Integrating (1.2.3), we get

$$\int \frac{dc}{c} = \frac{-k}{V}\int dt$$

$$\begin{aligned}
\log c &= \frac{-k}{V}t + \log D \\
\log c - \log D &= \frac{-k}{V}t \\
\log(\frac{c}{D}) &= \frac{-k}{V}t \\
\frac{c}{D} &= e^{\frac{-k}{V}t} \\
c(t) &= D\ exp\left(\frac{-k}{V}t\right)
\end{aligned}$$

Then, we have

$$c(t) = D\exp\left(-\frac{k}{V}t\right), 0 \leqslant t < T$$

At time $T$, the residue of the first dose is $D\exp\left(-\frac{k}{V}T\right)$ and now another dose $D$ is given so that we get

$$\begin{aligned}
c(t) &= \left(D\exp\left(-\frac{k}{V}T\right) + D\right)\exp\left(-\frac{k}{V}(t-T)\right), \\
&= D\exp\left(-\frac{k}{V}t\right) + D\exp\left(-\frac{k}{V}(t-T)\right),
\end{aligned}$$

$$T \leqslant t < 2T$$

The first term gives the residual of the first dose and the second term gives the residual of the second dose. Proceeding in the same way, we get after $n$ doses have been given

$$\begin{aligned}
c(t) =& D\exp\left(-\frac{k}{V}t\right) + D\exp\left(-\frac{k}{V}(t-T)\right) \\
&+ D\exp\left(-\frac{k}{V}(t-2T)\right) + \ldots + D\exp\left(-\frac{k}{V}(t-\overline{n-1}T)\right) \\
=& D\exp\left(-\frac{k}{V}t\right)\left(1+\exp\left(\frac{k}{V}T\right) + \exp\left(\frac{2k}{V}T\right)\right. \\
&\left.+\ldots+\exp\left((n-1)\frac{k}{V}T\right)\right) \\
=& D\exp\left(-\frac{k}{V}t\right)\frac{\exp\left(n\frac{k}{V}T\right)-1}{\exp\left(\frac{k}{V}T\right)-1}, (n-1)T \leqslant t < nT \\
c(nT-0) =& D\frac{1-\exp\left(-\frac{k}{V}nT\right)}{\exp\left(\frac{kT}{V}\right)-1} \\
c(nT+0) =& D\frac{\exp\left(\frac{kT}{V}\right)-\exp\left(-\frac{k}{V}nT\right)}{\exp\left(\frac{kT}{V}\right)-1}
\end{aligned}$$

Thus the concentration never exceeds $D/\left(1-\exp\left(-\frac{kT}{V}\right)\right)$. The graph of $c(t)$ is shown in Figure 1.4.

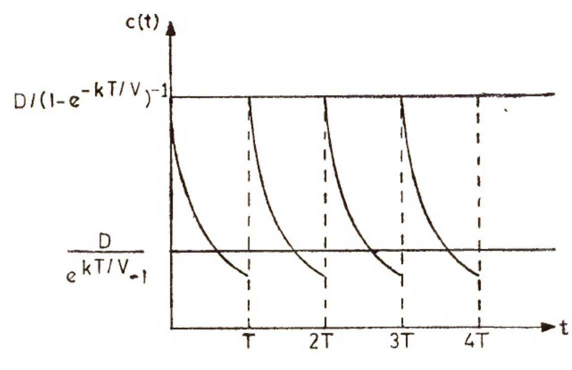



Figure 1.5

Thus in each interval, concentration decreases.

In any interval, the concentration is maximum at the beginning of this interval and thus maximum concentration at the beginning of an interval goes on increasing as the number of intervals increases, but the maximum value is always below $D/\left(1-e^{-kT/V}\right)$.

The minimum value in an interval occurs at the end of each interval. This also increases, but it lies below $D/(\exp(kT/V)-1)$.

The concentration curve is piecewise continuous and has points of discontinuity at $T, 2T, 3T, \ldots$

By injecting glucose or penicillin in blood and fitting curve $c(t)$ to the data, we can estimate the value of $k$ and $V$. In particular this gives a method for finding the volume of blood in the human body. □

**Remark 1.4.1. The Case of a Succession of Compartments**

*Let a solution with concentration $c(t)$ of a solute pass successively into $n$ tanks in which the initial concentrations of the solution are $c_1(0), c_2(0), \ldots, c_n(0)$.*

*The rates of inflow in each tank is the same as the rate of outflow from the tank. We have to find the concentrations $c_1(t), c_2(t) \ldots c_n(t)$ at time $t$. We get the equations*

$$V\frac{dc_1}{dt} = Rc - Rc_1$$
$$V\frac{dc_2}{dt} = Rc_1 - Rc_2$$
$$\ldots\ldots$$
$$V\frac{dc_n}{dt} = Rc_{n-1} - Rc_n$$

*By solving the first of these equations, we get $c_1(t)$. Substituting the value of $c_1(t)$ and proceeding in the same way, we can find $c_3(t), \ldots, c_n(t)$.*

## Let us sum up:

- A simple compartmental model.
- A method for finding the volume of blood in the human body and its remarks.

## Check your Progress:

1. A patient was given 0.5 micro-Curies (uci) of a type of iodine. Two hours later 0.5 uci had been taken up by his thyroid. How much would have been taken by the

thyroid in two hours if he had been given 15uci?

2. Explain compartmental models.

## 1.5 Dynamics Problems

### 1.5.1 Simple Harmonic Motion

Let a particle travel a distance $x$ in time $t$ in a straight line, then its velocity $v$ is given by $dx/dt$ and its acceleration is given by

$$dv/dt = (dv/dx)(dx/dt) = vdv/dx = d^2x/dt^2$$

Here a particle moves in a straight line in such a manner that its acceleration is always proportional to its distance from the origin and is always directed towards the origin, so that

$$\begin{aligned} v\frac{dv}{dx} &= -\mu x \qquad (1.5.14) \\ vdvdx &= -\mu x dx \end{aligned}$$

Integrating

$$v^2 = -\mu x^2 + A,$$

where the particle is initially at rest at $x = a$ (v=0).

$$\begin{aligned} A &= \mu a^2 \\ \implies \quad v^2 &= \mu \left(a^2 - x^2\right), \end{aligned}$$

Equation (1.5.15) gives

$$\frac{dx}{dt} = -\sqrt{\mu}\sqrt{a^2 - x^2}$$

We take the negative sign since velocity increases as $x$ decreases (Figure 1.6).

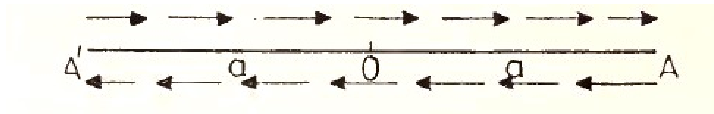



Figure 1.6

Integrating again and using the condition that at $t = 0, x = a$

$$x(t) = a \cos \sqrt{\mu} t$$

so that

$$v(t) = -a\sqrt{\mu} \sin \sqrt{\mu} t,$$

Thus in simple harmonic motion, both displacement and velocity are periodic functions with period $2\pi/\sqrt{\mu}$.

The particle starts from $A$ with zero velocity and moves towards 0 with increasing velocity and reaches 0 at time $\pi/2\sqrt{\mu}$ with velocity $\sqrt{\mu}a$.

It continue to move in the same direction, but now with decreasing velocity till it reaches $A'$ $(0A' = a)$ where its velocity is again zero.

It then begins moving towards 0 with increasing velocity and reaches 0 with velocity $\sqrt{\mu}a$ and again comes to rest at $A$ after a total time period $2\pi/\sqrt{\mu}$. The periodic motion then repeats itself.

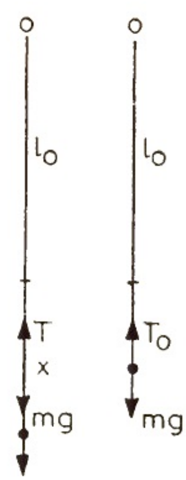



Figure 1.7

As one example of SHM, consider a particle of mass $m$ attached to one end of a perfectly elastic string, the other end of which is attached to a fixed point 0 (Figure

1.7). The particle moves under gravity in vacuum.

Let $l_0$ be the natural length of the string and let $a$ be its extension when the particle is in equilibrium so that by Hooke's law

$$mg = T_0 = \lambda \frac{a}{l_0}$$

where $\lambda$ is the coefficient of elasticity. Now let the string be further stretched a distance $c$ and then the mass be left free. The equation of motion which states that mass $x$ acceleration in any direction = force

On the particle in that direction, gives

$$\begin{aligned} mv\frac{dv}{dx} &= mg - T \\ &= \lambda\frac{a}{l_0} - \lambda\frac{a+x}{l_0} \\ &= -\frac{\lambda x}{l_0} \end{aligned}$$

or

$$v\frac{dv}{dx} = \frac{\lambda}{m}\frac{x}{l_0} = -\frac{gx}{a}$$

which gives a simple harmonic motion with time period $2\pi\sqrt{\frac{a}{g}}$. □

### 1.5.1 Motion Under Gravity in a Resisting Medium

A particle falls under gravity in a medium in which the resistance is proportional to the velocity. The equation of motion is

$$m\frac{dv}{dt} = mg - mkv$$

or

$$\begin{aligned} \frac{dv}{dt} &= g - kv \\ \frac{dv}{dt} &= k(\frac{g}{k} - v) \\ \frac{dv}{dt} &= k(V - v); \quad V = \frac{g}{k} \\ \frac{dv}{V - v} &= kdt \end{aligned}$$

Integrating

$$
\begin{aligned}
\frac{dv}{V-v} &= kdt \\
\log(V-v) &= kt + logA \\
V - v &= Ae^{-kt} \qquad (1.5.15)
\end{aligned}
$$

If the particle starts from rest with zero velocity, i.e v=0 when t=0. Equation (1.5.15) gives

$$
\begin{aligned}
V &= Ae^{0} \\
A &= V \\
\therefore\ v &= V\left(1 - e^{-kt}\right)
\end{aligned}
$$

so that the velocity goes on increasing and approaches the limiting velocity $g/k$ as $t \to \infty$. Replacing $v$ by $dx/dt$, we get

$$
\frac{dx}{dt} = V\left(1 - e^{-kt}\right)
$$

Integrating and using $x = 0$ when $t = 0$, we get

$$
\begin{aligned}
\int dx &= V\left(1 - e^{-kt}\right)\int dt \\
x(t) &= V(t - \frac{e^{-kt}}{-k}) + A \\
0 &= V(0 + \frac{1}{k}) + A \quad \text{(x=0, t=0)} \\
A &= -\frac{V}{k} \text{Thus, } x \qquad = Vt + \frac{Ve^{-kt}}{k} - \frac{V}{k}
\end{aligned}
$$

□

### 1.5.1 Motion of a Rocket

As a first idealisation, we neglect both gravity and air resistance. A rocket moves forward because of the large supersonic velocity with which gases produced by the burning of the fuel inside the rocket come out of the converging-diverging nozzle of the rocket (Figure 1.8).

Let $m(t)$ be the mass of the rocket at time $t$ and let it move forward with velocity

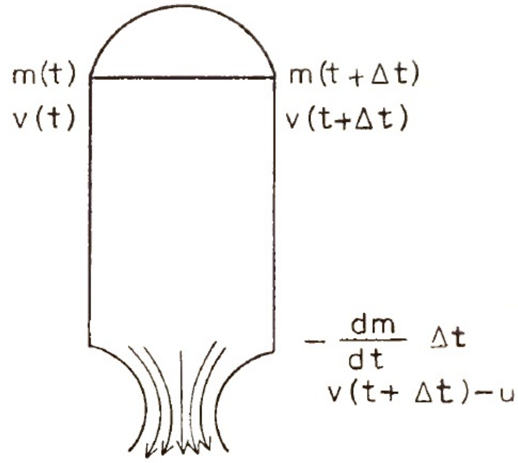



Figure 1.8

$v(t)$ so that the momentum at time $t$ is $m(t)v(t)$.

In the interval of time $(t, t+\Delta t)$, the mass of the rocket becomes

$$m(t+\Delta t) = m(t) + \frac{dm}{dt}\Delta t + o(\Delta t)$$

Since the rocket is losing mass, $dm/dt$ is negative and the mass of gases $-dm/dt\Delta t$ moves with velocity $u$ relative to the rocket, i.e. with a velocity $v(t+\Delta t) - u$ relative to the earth so that the total momentum of the rocket and the gases at time $t + \Delta t$ is

$$m(t+\Delta t)v(t+\Delta t) - \frac{dm}{dt}\Delta t(v(t+\Delta t) - u)$$

Since we are neglecting air resistance and gravity, there is no external force on the rocket and as such the momentum is conserved, giving the equation

$$m(t)v(t) = \left(m(t) + \frac{dm}{dt}\Delta t\right)\left(v(t) + \frac{dv}{dt}\Delta t\right) - \frac{dm}{dt}\Delta t(v-u) + o(\Delta t)^2$$

Simplifying and dividing by $\Delta t$ and proceeding to the limit as $\Delta t \to 0$, we get

$$\begin{aligned} m(t)v(t) =& m(t)v(t) + \frac{dm}{dt}v(t)\Delta t + m(t)\frac{dv}{dt}\Delta t + \frac{dm}{dt}v(t)\frac{dv}{dt}(\Delta t)^2 \\ &- \frac{dm}{dt}v(t)\Delta t + \frac{dm}{dt}u(t)\Delta t + 0(\Delta t)^2 \\ m(t)\frac{dv}{dt} =& -u\frac{dm}{dt} \quad \text{(By dividing } \Delta \text{ t and } \Delta t \to 0 \text{ )} \end{aligned}$$

or

$$\frac{dm}{m} = -\frac{1}{u}dv$$

By integrating and assuming that the rocket starts with zero velocity,

$$\begin{aligned}
\log m(t) &= -\frac{1}{u}v(t) + \log A \\
\log m(0) &= -\frac{1}{u}(0) + \log A \quad \text{(t=0, v=0)} \\
\log A &= \log m(0)
\end{aligned}$$

Thus

$$\log \frac{m(t)}{m(0)} = -\frac{v(t)}{u} \tag{1.5.16}$$

As the fuel burns, the mass of the rocket decreases.

Initially the mass of the rocket $= m_P + m_F + x_S$ when $m_P$ is the mass of the pay-load, $m_F$ is the mass of the fuel and $m_S$ is the mass of the structure.

When the fuel is completely burnt out, $m_F$ becomes zero and if $v_B$ is the velocity of the rocket at this stage, when the fuel is all burnt, then (1.5.16) gives

$$v_B = u \ln \frac{m_P + m_F + m_S}{m_P + m_S} = u \ln \left(1 + \frac{m_F}{m_P + m_S}\right)$$

This is the maximum velocity that the rocket can attain and it depends on the velocity $u$ of efflux of gases and the ratio $m_F / (m_P + m_S)$.

The larger the values of $u$ and $m_F / (m_P + m_S)$, the larger will be the maximum velocity attained.

For the best modern fuels and structural materials, the maximum velocity this gives is abount 7 km/sec.

In practice it would be much less since we have neglected air resistance and gravity, both of which tend to reduce the velocity.

However if a rocket is to place a satellite in orbit, we require a velocity of more than 7 km/sec.

The problem can be overcome by using the concept of multi-stage rockets.

The fuel may be carried in a number of containers and when the fuel of a container is burnt up, the container is thrown away, so that the rocket has not to carry any dead weight.

Thus in a three-stage rocket, let $m_{F_1}, m_{F_2}, m_{F_3}$ by the masses of the fuels and $m_{S_1}, m_{S_2}, m_{S_3}$ be the three corresponding masses of containers, then velocity at the end of the first stage is

$$v_1 = u \ln \frac{m_P + m_{F_1} + m_{S_1} + m_{F_2} + m_{S_2} + m_{F_3} + m_{S_3}}{m_P + m_{F_2} + m_{S_2} + m_{F_3} + m_{S_3}}$$

At the end the second stage, the velocity is

$$v_2 = v_1 + u \ln \frac{m_P + m_{F_2} + m_{F_3} + m_{S_3}}{m_P + m_{F_3} + m_{S_3}}$$

and at the end of the third stage, the velocity

$$v_3 = v_2 + u \ln \frac{m_P + m_{F_3}}{m_P}$$

In this way, a much larger velocity is obtained than can be obtained by a single-stage rocket.

## Let us sum up:

- Simple harmonic motion with time period.
- Gravitational motion in a resistive medium.
- Motion of a rocket.

## Check your progress:

1. What happens to the mass of the rocket when the fuel burns?
2. Explain the Periodic motion of the simple harmonic motion.

## 1.6 Simple Geometrical Problems

Many geometrical entities can be expressed in terms of derivatives and as such relations between these entities can give rise to differential equations whose solution will give us a family of curves for which the given relation between geometrical entities is satisfied.

(i) Find curves for which tangent at a point is always perpendicular to the line joining the point to the origin.

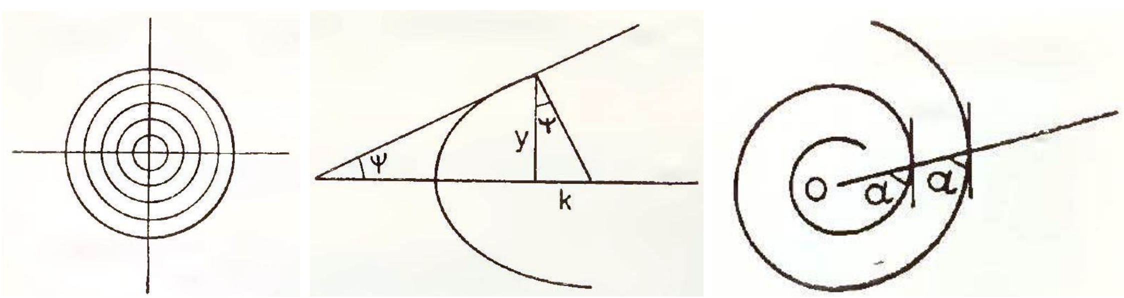
The slope of the tangent is $dy/dx$ and the slope of line joining the point $(x, y)$ to the origin is $y/x$ and since these lines are given to be orthogonal

$$\frac{dy}{dx} = -\frac{x}{y}$$

Integrating

$$\int ydy = -\int xdx$$
$$y^2 = -x^2 + a^2$$
$$x^2 + y^2 = a^2$$

which represents a family of concentric circle.




(a) (b) (c)

Figure 1.9

(ii) Find curves for which the projection of the normal on the $x$-axis is of constant length.

This condition gives

$$y\frac{dy}{dx} = k$$

Integrating

$$y^2 = 2kx + A,$$

which represents a family of parabolas, all with the same axis and same length of latus rectum.

(iii) Find curves for which tangent makes a constant angle with the radius vector.

Here it is convenient to use polar coordinates and the conditions of the problem gives

$$r\frac{d\theta}{dr} = \tan\alpha$$

Integrating

$$\int \frac{dr}{r} = \cot\alpha \int d\theta$$

$$\log r = \theta\cot\alpha + \log A$$

$$\log r - \log A = \theta\cot\alpha$$

$$\log\frac{r}{A} = \theta\cot\alpha$$

$$\frac{r}{A} = e^{\theta\cot\alpha}$$

$$r = Ae^{\theta\cot\alpha}$$

which represents a family of equiangular spirals. □

## Let us sum up:

- Simple geometrical problems in finding the curves for tangent at a point.
- Finding the curves for projection of the normal on the x-axis.
- Finding the curves for the tangent making a constant angle.

## Check your progress:

1. Find the curves of the tangent at a point is always perpendicular to the line joining the point to the origin.
2. Find the curves for which tangent makes a constant angle with the radius vector

## 1.7 Orthogonal Trajections

Let

$$f(x, y, a) = 0 \tag{1.7.17}$$

represent a family of curves, one curve for each value of the parameters $a$. Differentiating (1.7.17), we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \tag{1.7.18}$$

Eliminating $a$ between (1.7.17) and (1.7.18), we get a differential equation of the first order

$$\varphi\left(x, y, \frac{dy}{dx}\right) = 0 \tag{1.7.19}$$

of which (1.7.17) is the general solution. Now we want a family of curves cutting every member of (1.7.17) at right angle at all points of intersection.

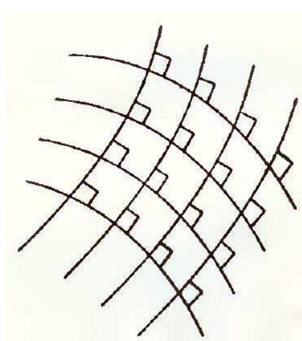


Figure 1.10

At a point of intersection of the two curves, $x, y$ are the same but the slope of the second curve is negative reciprocal of the slope of the first curve.

As such differential equation of the family of orthogonal trajectories is

$$\varphi\left(x, y, -\frac{1}{dy/dx}\right) = 0 \tag{1.7.20}$$

Integrating (1.7.20), we get

$$g(x, y, b) = 0$$

which gives the orthogonal trajectories of the family (1.7.17).

(i) Let the original family be $y = mx$, when $m$ is a parameter then

$$dy/dx = m$$

and eliminating $m$, we get the differential equation of this concurrent family of straight lines as

$$\frac{y}{x} = \frac{dy}{dx}$$

To get the orthogonal trajectories, we replace $dy/dx$ by $-1/(dy/dx)$ to get

$$\frac{y}{x} = -\frac{1}{dy/dx}$$

Integrating

$$\int y dy = -\int x dx$$

$$y^2 = -x^2 + a^2$$

$$x^2 + y^2 = a^2$$

$$x^2 + y^2 = a^2 \qquad (1.7.21)$$

which gives the orthogonal trajectories as concentric circles (Figure 1.3.9a).

(ii) Find the orthogonal trajectories of the family of confocal conics.

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1 \qquad (1.7.22)$$

where $\lambda$ is a parameter. Differentiating, we get

$$\frac{x}{a^2 + \lambda} + \frac{y}{b^2 + \lambda} \frac{dy}{dx} = 0 \qquad (1.7.23)$$

Eliminating $\lambda$ between (1.7.22) and (1.7.23), we get

$$(xp - y)(x + py) = p\left(a^2 - b^2\right); \quad p = \frac{dy}{dx} \tag{1.7.24}$$

To get the orthogonal trajectories, we replace $p$ by $-\frac{1}{p}$ to get

$$\left(-\frac{x}{p} - y\right)\left(x - \frac{y}{p}\right) = -\frac{1}{p}\left(a^2 - b^2\right) \tag{1.7.25}$$

or

$$(xp - y)(x + py) = p\left(a^2 - b^2\right) \tag{1.7.26}$$

However (1.7.24) and (1.7.26) are identical. As such the family of confocal conics is self-orthogonal, i.e. for every conic of the family, there is another with same focii which cuts it at right angles.

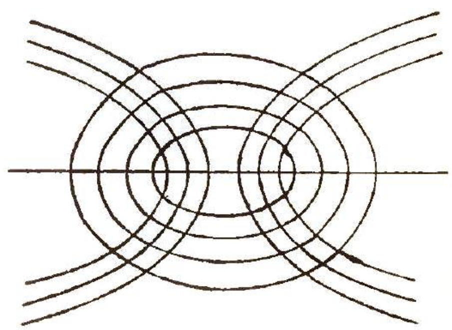


Figure 1.11

One family consists of confocal ellipses and the other consists of confocal hyperbolas with the same focii (Figure 1.11).

(iii) In polar coordinates after getting the differential equation of the family of curves, we have to replace $r\frac{d\theta}{dr}$ by $-1/\left(r\frac{d\theta}{dr}\right)$ and then integrate the resulting differential equation.

Then if the original family is

$$r = 2a\cos\theta, \tag{1.7.27}$$

with $a > 0$ as a parameter, its differential equation is obtained by eliminating $a$ between (1.7.27) and

$$\frac{dr}{d\theta} = -2a\sin\theta$$

to get

$$r\frac{d\theta}{dr} = -\cot\theta$$

Replacing $r\frac{d\theta}{dr}$ by $-\left(r\frac{d\theta}{dr}\right)^{-1}$, we get

$$r\frac{d\theta}{dr} = \tan\theta$$

Integrating we get

$$r = 2b\sin\theta$$

The orthogonal trajectories are shown in Figure 1.12.

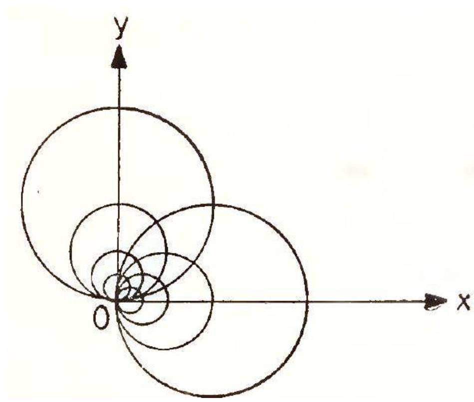



Figure 1.12

The circles of both families pass through the origin, but while the centre of one family lie on $x$-axis, the centres of the orthogonal family lie on $y$-axis.

## Let us sum up:

- The orthogonal trajectories with the family of curves consisting confocal ellipses .
- Other family consists of confocal hyberbolas with the same focii.
- The circles of the families passes through the origin.

## Check your progress:

1. Find the orthogonal trajectories of the differential equation $\varphi(x, y, -\frac{1}{dy/dx}) = 0$.

2. Find the orthogonal trajectories of the family of confocal conics.

## Summary:

In this unit, we have modelled and analyzed the population growth and decay models that change over time. In addition, to studied the spread of technological innovations and infectious diseases. Further, we discussed the basics of the law of mass action: chemical reactions. Finally, we have analyzed the dynamical problems like simple harmonic motion, motion under gravity in a resisting medium, motion of a rocket and orthogonal trajectory.

## Glossary:

Population growth model, Immigration, Emigration, Radio-active decay, Logistic law, Compartmental model, Orthogonal trajectory.

## Self Assessment questions

1. What are changes explained in the population size that depends on birth and death rate?

2. Explain decay models.

3. Explain model used in medicine is that the rate of growth of a tumor is proportional to the size of the tumor.Find the General solution.

4. Find the volume of Blood in the human body?

5. Explain the orthogonal trajectories when the circles of families passes through the origin.

## Exercises

1. Suppose the population of the world now is 4 billion and its doubling period is 35 years, what will be the population of the world after 350 years, 700 years, 1050 years? If the surface area of the earth is 1,860,000 billion square feet, how much space would each person get after 1050 years?

2. Find the relation between doubling, tripling and quadrupling times for a population.

3. Substances X and Y combine in the ratio 2 : 3 to form Z. When 45 grams of X and 60 grams of Y are mixed together, 50 gms of Z are formed in 5 minutes. How many grams of Z will be found in 210 minutes? How much time will it like to get 70 gms of Z?

4. Show that he logistic model can be written as $\frac{1}{N}\frac{dN}{dt} = r\frac{K-N}{N}$ . Deduce that $K$ is the limiting size of the population and the average rate of growth is proportional to the fraction by which the population is unsaturated.

5. Let

$$G(t)$$

be the amount of glucose present in the blood-stream of a patient at time t. Assuming that the glucose is injected into the blood stream at a constant rate of $C$ grams per minute, and at the same time is converted and removed from the blood stream at a rate proportional to the amount of glucose present, find the amount $G(t)$ at any time t. If $G(0) = G_0$, what is the equilibrium level of glucose in the blood stream?

6. Discuss the motion of a rocket when gravity is taken into account.

7. Find a family of curves such that for each curve, the length of the tangent intercepted between the axes is of constant length. Draw the curves

# Answers for check your progress

### Section 1.2

1. The population size remains constant.

2. The planting of new plants will correspond to immigration and cutting of trees will correspond to emigration.

### Section 1.3

1. Refer section 1.3.1

2. If $A$ and $B$ are the initial amounts of two substances, we get

$$\frac{dz}{dt} = k(A - \frac{az}{a+b})(B - \frac{bz}{a+b}).$$

   This a non linear differential equation for a second order reaction.

### Section 1.4

1. 0.5 microcuries

2. Refer section 1.4.1

### Section 1.5

1. As the fuel burns, the mass of the rocket decreases.

2. A particle starts from $A$ with zero velocity and moves towards 0 with increasibg velocity and reaches 0 at time $\pi/2\sqrt{\mu}$ with velocity $\sqrt{\mu}a$. It continue to move in the same direction, but now with decreasing velocity till it reaches where its velocity is again zero. If it begins moving towards 0 with increasing velocity and reaches 0 with velocity $\sqrt{\mu}a$ and comes to reat ar $A$ after a total time period $2\pi/\sqrt{\mu}$. The periodic motion then repeats itself.

**Section 1.6**

1. It is a family of concentric circle.

2. It is a family if equiangular spirals.

**Section 1.7**

1. The orthogonal trajectories as concentric circles .

2. One family consists of confocal ellipses and the other consists of confocal hyperbolas with the same focii.

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# UNIT - 2

# Unit 2

# Mathematical Modelling Through Systems of First-Order Ordinary Differential Equations

## Objectives:

- To model and analyze population dynamics models.
- Understand the spread of diseases through epidemic models and infectious disease models.
- To discuss the compartment model, economics, medicine and arms race.
- To solve simple problems in first-order ordinary differential equations.

## 2.1 Mathematical Modelling in Population Dynamics

### 2.1.1 Prey-Predator Model

Let $x(t), y(t)$ be the populations of the prey and predator species at time $t$. We assume that

(i) if there are no predators, the prey species will grow at a rate proportional to the population of the prey species,

(ii) if there are no prey, the predator species will decline at a rate proportional to the population of the predator species,

(iii) the presence of both predators and preys is beneficial to growth of predator species and is harmful to growth of prey species. More specifically the predator species increases and the prey species decreases at rates proportional to the product of the two populations.

These assumptions give the systems of non-linear first order ordinary differential equations

$$\frac{dx}{dt} = ax - bxy = x(a - by), a, b > 0 \quad (2.1.1)$$
$$\frac{dy}{dt} = -py + qxy = -y(p - qx), p, q > 0 \quad (2.1.2)$$

Now $dx/dt, dy/dt$ both vanish if

$$x = x_e = \frac{p}{q}, \quad y = y_e = \frac{a}{b}.$$

If the initial populations of prey and predator species are $p/q$ and $a/b$ respectively, the populations will not change with time. These are the equilibrium sizes of the populations of the two species.

Of course $x = 0, y = 0$ also gives another equilibrium position.

From (2.1.1) and (2.1.2),

$$\frac{dy}{dx} = -\frac{y(p - qx)}{x(a - by)}$$

or

$$\frac{a - by}{y} dy = -\frac{p - qx}{x} dx; \quad x_0 = x(0), \quad y_0 = y(0)$$
$$\frac{ady}{y} - bydy = \frac{-pdx}{x} + qxdx$$

Integrating

$$\int \frac{ady}{y} - \int bdy = \int \frac{-pdx}{x} + \int qdx$$

$$a \ln y(t) - by(t) = -p \ln x(t) + qx(t) + A$$

$$a \ln y(0) - by(0) = -p \ln x(0) + qx(0) + A \quad (\text{since } x_0 = x(0), y_0 = y(0))$$

$$\therefore \; A = a \ln y(0) - by(0) + p \ln x(0) - qx(0)$$

Substituting A values, we get

$$a \ln y(t) - by(t) = -p \ln x(t) + qx(t) + a \ln y(0) - by(0) + p \ln x(0) - qx(0)$$

$$a \ln y(t) - a \ln y(0) + p \ln x(t) - p \ln x(0) = by(t) - by(0) + qx(t) - qx(0)$$

$$a \ln \frac{y}{y_0} + p \ln \frac{x}{x_0} = b\,(y - y_0) + q\,(x - x_0)$$

Thus through every point of the first quadrant of the $x - y$ plane, there is a unique trajectory. No two trajectories can intersect, since intersection will imply two different slopes at the same point.

If we start with $(0,0)$ or $(p/q, a/b)$, we get point trajectories.

If we start with $x = x_0, y = y_0$, from (2.28) and (2.29), we find that $x$ increases while $y$ remains zero.

Similarly if we start with $x = 0, y = y_0$, we find that $x$ remains zero while $y$ decreases.

Thus positive axes of $x$ and $y$ give two line trajectories (Figure 2.1).

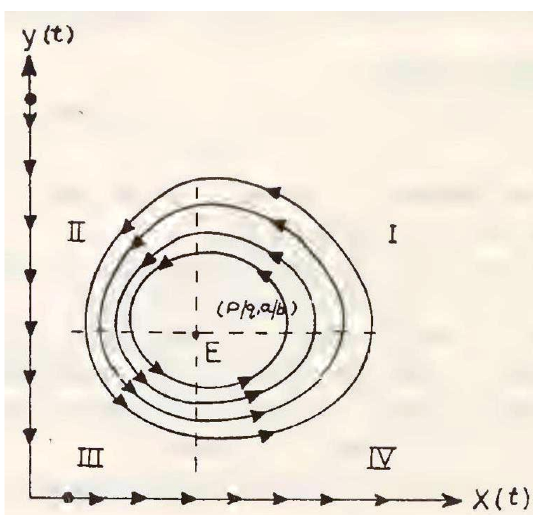


Figure 2.1: Prey-Predator Model

Since no two trajectories intersect, no trajectory starting from a point situated within the first quadrant will intersect the $x$-axis and $y$-axis trajectories.

Thus all trajectories corresponding to positive initial populations will lie strictly within the first quadrant. Thus if the initial populations are positive, the populations will be always positive. If the population of one (or both) species is initially zero, it will always remain zero.

The lines through $(p/q, a/b)$ parallel to the axes of coordinates divide the first quadrant into four parts I, II, III and IV. Using (2.1.1), (2.1.2), we find that

$$\begin{array}{llll} \text{in I ,} & dx/dt < 0, & dy/dt > 0, & dy/dx < 0 \\ \text{in II,} & dx/dt < 0, & dy/dt < 0, & dy/dx > 0 \\ \text{in III,} & dx/dt > 0, & dy/dt < 0, & dy/dx < 0 \\ \text{in IV,} & dx/dt > 0, & dy/dt > 0, & dy/dx > 0 \end{array}$$

This give the direction field at all points as shown in Figure 13. Each trajectory is a closed convex curve. These trajectories appear relatively cramped near the axes.

In I and II, prey species decreases and in III and IV, it increases. Similarly in IV and I, predator species increases and in II and III, it decreases.

After a certain period, both species return to their original sizes and thus both species sizes vary periodically with time. □

### 2.1.2 Competition Models

Let $x(t)$ and $y(t)$ be the populations of two species competing for the same resources, then each species grows in the absence of the other species, and the rate of growth of each species decreases due to the presence of the other species.

This gives the system of differential equations

$$\frac{dx}{dt} = ax - bxy = bx\left(\frac{a}{b} - y\right); \quad a > 0, \quad b > 0 \tag{2.1.3}$$

$$\frac{dy}{dt} = py - qxy = y(p - qx) = qy\left(\frac{p}{q} - x\right); \quad p > 0, \quad q > 0 \tag{2.1.4}$$

There are two equilibrium positions viz. $(0, 0)$ and $(p/q, a/b)$. There are two point trajectories viz. $(0, 0)$ and $(p/q, a/b)$ and there are two line trajectories viz. $x = 0$ and $y = 0$.

$$\text{In I}\quad dx/dt < 0,\quad dy/dt < 0,\quad dy/dx > 0$$

$$\text{In II}\quad dx/dt < 0,\quad dy/dt > 0,\quad dy/dx < 0$$

$$\text{In III}\quad dx/dt > 0,\quad dy/dt > 0,\quad dy/dx > 0$$

$$\text{In IV}\quad dx/dt > 0,\quad dy/dt < 0,\quad dy/dx < 0$$

This gives the direction field as shown in Figure 2.2. From (2.1.3) and (2.1.4)

$$\frac{dy}{dx} = \frac{y(p-qx)}{x(a-by)} \quad \text{or} \quad \frac{a-by}{y}dy = \frac{p-qx}{x}dx$$

$$\frac{a-by}{y}dy = \frac{p-qx}{x}dx; \quad x_0 = x(0), \quad y_0 = y(0)$$

$$\frac{ady}{y} + \frac{pdx}{x} = bydy - qxdx$$

Integrating

$$\int \frac{ady}{y} - \int \frac{pdx}{x} = \int bdy - \int qdx$$

$$a\ln y(t) - p\ln x(t) = by(t) - qx(t) + A$$

$$a\ln y(0) - p\ln x(0) = by(0) - qx(0) + A \quad (\text{since } x_0 = x(0), y_0 = y(0))$$

$$\therefore\ A = a\ln y(0) - p\ln x(0) - by(0) + qx(0)$$

Substituting A values, we get

$$a\ln y(t) - p\ln x(t) = by(t) - qx(t) + a\ln y(0) - p\ln x(0) - by(0) + qx(0)$$

$$a\ln y(t) - a\ln y(0) - p\ln x(t) + p\ln x(0) = by(t) - by(0) - qx(t) + qx(0)$$

$$a\ln\frac{y}{y_0} - b\left(y - y_0\right) = p\ln\frac{x}{x_0} - q\left(x - x_0\right)$$

The trajectory which passes through $(p/q, a/b)$ is

$$a\ln\frac{by}{a} - by + a = p\ln\frac{qx}{p} - qx + p$$

If the initial populations correspond to the point $A$, ultimately the first species dies but and the second species increases in size to infinity.

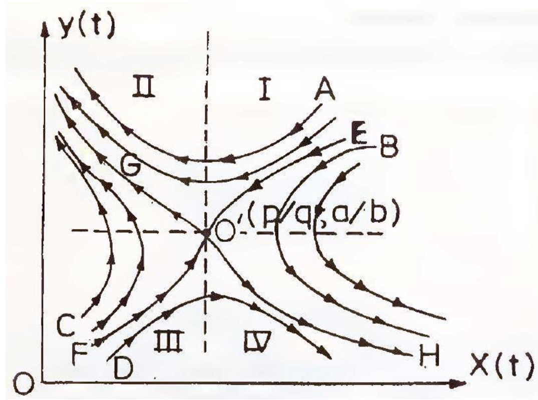


Figure 2.2: Competition Model

If the initial populations correspond to the point $B$, then ultimately the second species dies out and the first species tends to infinity.

Similarly if the initial populations correspond to point $C$, the first species dies out and the second species goes to infinity and if the initial populations correspond to point $D$, the second species dies out and the first species goes to infinity.

If the initial populations correspond to point $E$ or $F$, the species populations converge to equilibrium populations $p/q, a/b$ and if the initial population correspond to point $G, H$, the first and second species die out respectively.

Thus except when the initial populations correspond to points on curves $O'E$ and $O'F$, only one species will survive in the competition process and the species can coexist only when the initial population sizes correspond to points on the curve $EF$.

It is also interesting to note that while the initial populations corresponding to $A, E, B$ are quite close to one another, the ultimate behaviour of these populations are drastically different.

For populations starting at $A$, the second species alone survives, for populations starting at $B$, the first species alone survives, while for population starting at $E$, both species can coexist.

Thus a slight change in the initial population sizes can have a catastrophic effect on the ultimate behaviour. □

**Remark 2.1.1.** *It may also be noted that for both prey-predator and competition models, we have obtained a great deal of insight into the models without using the solution of these equations (2.1.1), (2.1.2) or (2.1.3), (2.1.4). By using numerical methods of integration* *with the help of computers, we can draw some typical trajectories in both cases and can get additional insight into the behaviour of these models.*

### 2.1.3 Multi-species Models

We can consider the model represented by the system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= a_1x_1 + b_{11}x_1^2 + b_{12}x_1x_2 + \ldots + b_{1n}x_1x_n \\ \frac{dx_2}{dt} &= a_2x_2 + b_{21}x_2x_1 + b_{22}x_2^2 + \ldots + b_{2n}x_2x_n \\ \frac{dx_n}{dt} &= a_nx_n + b_{n1}x_nx_1 + b_{n2}x_nx_2 + \ldots + b_{nn}x_n^2 \end{aligned} \tag{2.1.5}$$

Here $x_1(t), x_2(t), \ldots, x_n(t)$ represent the populations of the $n$ species.

Also $a_i$ is positive or negative according as the $i$ th species grows or decays in the absence of other species and $b_{ij}$ is positive or negative according as the $i$ th species benefits or is harmed by the presence of the $j$ th species.

In general $b_{ii}$ is negative since members of the $i$ th species also compete among themselves for limited resources.

We can find the positions of equilibrium by putting

$$\frac{dx_i}{dt} = 0 \quad \text{for} \quad i = 1, 2, \ldots, n$$

and solving the $n$ algebraic equations for $x_1, x_2, \ldots, x_n$.

We can also obtain all degenerate solutions in which one or more $x_i$ 's are zero, i.e. in which one or more species have disappeared and finally we have the equilibrium position in which all species can disappear.

If $x_{10}, x_{20}, \ldots, x_{n0}$ is an equilibrium position, we can discuss its local stability by substituting

$$x_1 = x_{10} + u_1, \quad x_2 = x_{20} + u_2, \ldots, \quad x_n = x_{n0} + u_n$$

in (2.1.5) and getting a system of linear differential equations

$$\frac{du_1}{dt} = c_{11}u_1 + c_{12}u_2 + \ldots + c_{1n}u_n$$
$$\frac{du_2}{dt} = c_{21}u_1 + c_{22}u_2 + \ldots + c_{2n}u_n$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{du_n}{dt} = c_{n1}u_1 + c_{n2}u_2 + \ldots\ldots + c_{nn}u_n$$

by neglecting squares, products and higher powers of $u_i$ 's.

We can try the solutions $u_1 = A_1 e^{\lambda_t}, u_2 = A_2 e^{\lambda_t}, \ldots, u_n = A_n e^{\lambda_t}$ to get

$$\begin{vmatrix} c_{11}-\lambda & c_{12} & c_{13} & \ldots & c_{1n} \\ c_{21} & c_{22}-\lambda & c_{23} & \ldots & c_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{n1} & c_{n2} & c_{n3} & \ldots & c_{nn}-\lambda \end{vmatrix} = 0$$

Thus the equilibrium position would be stable if the real parts of all the eigenvalues of the matrix $[c_{ij}]$ are negative.

The conditions for this are given by Routh-Hurwitz criterion which states that all the roots of

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0, \quad a_0 > 0 \tag{2.1.6}$$

will have negative real parts if and only if $T_0, T_1, T_2, \ldots$ are positive where

$$T_0 = a_0, \quad T_1 = a_1, \quad T_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \quad T_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}$$
$$T_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix}$$

This is true if and only if $a_i$ and either all even-numbered $T_k$ or all oddnumbered $T_k$ are positive.

Alternatively (2.1.6) will have all roots with negative real parts iff this is true for the $(n-1)$ th degree equation

$$a_1x^{n-1}+a_2x^{n-2}+a_3x^{n-3}+\ldots-\frac{a_0}{a_1}a_3x^{n-2}-\frac{a_0}{a_1}a_5x^{n-4}-\ldots=0$$

The above method will enable us to discuss only local stability of a position of equilibrium, i.e. this will decide that if the populations of different species are changed slightly from these equilibrium values, whether the population sizes will return to their original equilibrium values or not.

The problem of discussing the global stability i.e. of discussing whether the populations will return to these equilibrium values, whatever be the magnitudes of the disturbances, is a more difficult problem and it is possible to solve this problem in special cases only. □

### 2.1.4 Age-Structured Population Models

Let $x_1(t), x_2(t), \ldots, x_p(t)$ be the populations of the $p$ pre-reproductive age groups.

Let $x_{p+1}(t)$, $\ldots$, $x_{p+q}(t)$ be the populations of $q$ reproductive agegroups and let $x_{p+q+1}(t)$, $\ldots$, $x_{p+q+r}(t)$ be the populations of the $r$ post reproductive age-groups.

Let $b_{p+1}, b_{p+2}, \ldots, b_{p+q}$ be the birth rates in the $q$ reproductive age-groups, let $d_i$ be the death rates in the $i$ th age-group $(i=1,2,\ldots,p+q+r)$ and let $m_j$ be the rate of migration from the $j$ th age-group to the $(j+1)$ th age-group $(j=1,2,\ldots,p+q+r-1)$, then we get the system of differential equations

$$\begin{aligned}
&\frac{dx_1}{dt}=b_{p+1}x_{p+1}+\ldots+b_{p+q}x_{p+q}-(d_1+m_1)\,x_1\\
&\frac{dx_2}{dt}=m_1x_1-(d_2+m_2)\,x_2\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{dx_n}{dt}=m_{n-1}x_{n-1}-d_nx_n;\quad n=p+q+r,\quad \text{or}
\end{aligned}$$

$$\frac{d}{dt}\begin{bmatrix}x_1(t)\\x_2(t)\\\vdots\\x_n(t)\end{bmatrix}=\begin{bmatrix}-(d_1+m_1)&0&\ldots&b_{p+1}&\ldots&b_{p+q}&\ldots&0&0\\m_1&-(d_2+m_2)&\ldots&0&\ldots&0&\ldots&0&0\\0&m_2&\ldots&0&\ldots&0&\ldots&0&0\\\ldots&\ldots&\ldots&\ldots&\ldots&\ldots&\ldots&\ldots&\ldots\\\ldots&\ldots&\ldots&\ldots&\ldots&0&\ldots&m_{n-1}&-d_n\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\\vdots\\x_n(t)\end{bmatrix}\tag{2.1.7}$$

or

$$\frac{dX}{dt} = AX(t),$$

where A is a matrix, all of whose diagonal elements are negative, all of whose main subdiagonal elements are positive, q other elements of the first row are positive and all other elements are zero.

Equation (2.1.7) has the solution

$$X(t) = exp(At)X(0)$$

□

## Let us sum up:

- Prey-predator model.
- Competition models.
- Competition models.
- Age-structured population models.

## Check your progress:

1. What is the basic dynamics of Prey predator model?
2. What is the result of when no two trajectories can intersect?
3. What are the equilibrium points of the differential equations of the compartmental models?
4. What is the result of the Multispecies model ?

## 2.2 Mathematical Modelling of Epidemics

### 2.2.1 A Simple Epidemic Model

Let S(t) and I(t) be the number of susceptibles (i.e. those who can get a disease) and infected persons (i.e. those who have already got the disease).

Initially let there be $n$ susceptible and one infected person in the system so that

$$S(t) + I(t) = n + 1, \quad S(0) = n, \quad I(0) = 1$$

The number of infected persons grows at a rate proportional to the product of susceptible and infected persons and the number of susceptible persons decreases at the same rate so that we get the system of differential equations

$$\begin{aligned} \frac{dS}{dt} &= -\beta SI, \\ \frac{dI}{dt} &= \beta SI \end{aligned} \tag{2.2.8}$$

so that

$$\begin{aligned} \frac{dS}{dt} + \frac{dI}{dt} &= 0, \\ S(t) + I(t) &= \text{constant } = n + 1 \end{aligned} \tag{2.2.9}$$

By using (2.2.9) in (2.2.8), we get

$$\begin{aligned} &\frac{dS}{dt} = -\beta S(t)(n + 1 - S(t)) \\ &\frac{dI}{dt} = \beta I(t)(n + 1 - I(t)) \\ \implies \quad &\frac{dS}{dt} = -\beta S(n + 1 - S) \\ &\frac{dI}{dt} = \beta I(n + 1 - I) \end{aligned}$$

Integrating $\frac{dS}{dt} = -\beta S(n + 1 - S)$, we get

$$\int \frac{dS}{dt} = \int \beta S(t)(S(t) - (n + 1))$$

$$
\begin{aligned}
\int \frac{dS}{S(S-(n+1))} &= -\int \beta dt \\
\int \frac{1}{n+1}\Big[-\frac{1}{S}+\frac{1}{(S-(n+1))}\Big]dS &= \int \beta dt \\
\Big[-\int \frac{dS}{S}+\int \frac{dS}{(n+1-S)}\Big] &= (n+1)\int \beta dt \\
\Big[-\log S+\log(S-(n+1))\Big] &= (n+1)\beta t+\log C \\
\Big[-\log S+\log(S-(n+1))-\log C\Big] &= (n+1)\beta t \\
\log\Big[\frac{S-(n+1)}{CS}\Big] &= (n+1)\beta t \\
\frac{S-(n+1)}{CS} &= e^{(n+1)\beta t} \\
\frac{S-(n+1)}{S} &= Ce^{(n+1)\beta t} \qquad (2.2.10)
\end{aligned}
$$

By using the initial condition S(0) = n, we get

$$
\begin{aligned}
\frac{S(0)-(n+1)}{S(0)} &= Ce^{0} \\
\frac{n-n-1}{n} &= C \\
C &= -\frac{1}{n}
\end{aligned}
$$

Then equation (2.2.10) becomes

$$
\begin{aligned}
\frac{S-(n+1)}{S} &= -\frac{1}{n}e^{(n+1)\beta t} \\
n(S-(n+1)) &= -Se^{(n+1)\beta t} \\
nS-n(n+1) &= -Se^{(n+1)\beta t} \\
nS+Se^{(n+1)\beta t} &= n(n+1) \\
S(n+e^{(n+1)\beta t}) &= n(n+1) \\
S(t) &= \frac{n(n+1)}{n+e^{(n+1)\beta t}} \qquad (2.2.11) \\
Similarly,\ I(t) &= \frac{(n+1)e^{(n+1)\beta t}}{n+e^{(n+1)\beta t}} \qquad (2.2.12)
\end{aligned}
$$

From (2.2.11) and (2.2.12), we have

$$
S(t)=\frac{n(n+1)}{n+e^{(n+1)\beta t}}, \quad I(t)=\frac{(n+1)e^{(n+1)\beta t}}{n+e^{(n+1)\beta t}},
$$

so that

$$\mathrm{Lt}_{t\to\infty} S(t) = 0, \quad \mathrm{Lt}_{t\to\infty} I(t) = n + 1 \tag{2.2.13}$$

### 2.2.2 A Susceptible-Infected-Susceptible (SIS) Model

Here, a susceptible person can become infected at a rate proportional to $SI$ and an infected person can recover and become susceptible again at a rate $\gamma I$, so that

$$\frac{dS}{dt} = -\beta SI + \gamma I, \quad \frac{dI}{dt} = \beta SI - \gamma I, \tag{2.2.14}$$

which gives

$$\frac{dI}{dt} = (\beta(n+1) - \gamma)I - \beta I^2 \tag{2.2.15}$$

### 2.2.3 SIS Model with Constant Number of Carriers

Here infection is spread both by infectives and a constant number $C$ of carriers, so that (2.2.15) becomes

$$\begin{aligned}\frac{dI}{dt} &= \beta(I + C)S - \gamma I - \beta I^2 \\ &= \beta C(n+1) + \beta(n + 1 - \gamma/\beta)I - \beta I^2.\end{aligned}$$

### 2.2.4 Simple Epidence Model with Carriers

In this model, only carriers spread the disease and their number decreases exponentially with time as these are identified and eliminated, so that we get

$$\frac{dS}{dt} = -\beta S(t)C(t) + \gamma I(t), \tag{2.2.16}$$

$$\frac{dI}{dt} = \beta C(t)S(t) - \gamma I(t), \tag{2.2.17}$$

$$\frac{dC}{dt} = -\alpha C \tag{2.2.18}$$

so that

$$S(t) + I(t) = S_0 + I_0 = N(\text{ say }). \tag{2.2.19}$$

Integrating (2.2.18), we get

$$\begin{aligned}
\frac{dC}{C} &= -\alpha \int dt \\
\log C &= -\alpha t + \log A \\
\log \frac{C}{A} &= -\alpha t \\
\frac{C}{A} &= e^{-\alpha t} \\
C &= Ae^{-\alpha t} \\
C(t) &= C_0 \exp(-\alpha t) \;\; \text{(By taking } C(0) = C_0 \text{ and solving)} \qquad (2.2.20)
\end{aligned}$$

and by using (2.2.19), (2.2.20) in (2.2.17), we get

$$\frac{dI}{dt} = \beta C_0 N \exp(-\alpha t) - [\beta C_0 \exp(-\alpha t) + \gamma]\, I$$

### 2.2.5 Model with Removal

Here infected persons are removed by death or hospitalisation at a rate proportional to the number of infectives, so that the model is

$$\begin{aligned}
\frac{dS}{dt} = -\beta SI, \quad \frac{dI}{dt} &= \beta SI - \gamma I = \beta I \left( S - \frac{\gamma}{\beta} \right) \\
&= \beta I(S - \rho); \quad \rho = \frac{\gamma}{\beta}
\end{aligned}$$

with initial conditions

$$S(0) = S_0 > 0, \quad I(0) = I_0 > 0, \quad R(0) = R_0 = 0$$

$$S_0 + I_0 = N$$

### 2.2.6 Model with Removal and Immigration

We modify the above model to allow for the increase of susceptibles at a constant rate $\mu$ so that the model is

$$\frac{dS}{dt} = -\beta SI + \mu, \frac{dI}{dt} = \beta SI - \gamma I, \frac{dR}{dt} = \gamma I$$

□

## Let us sum up:

- A simple epidemic model.
- A susceptible-infected-susceptible (SIS) model.
- Simple epidemic model with carriers.
- Model with removal and immigration.

## Check your progress:

1. What are the types of epidemic models?
2. What is the difference between the Simple Epidence Model with Carriers and Model with removal?

## 2.3 Compartment Models

Pharmokinetics (also called drug kinetics or tracer kinetics or multi-compartment analysis) deals with the distribution of drugs, chemicals, tracers or radio-active substances among various compartments of the body where compartments are real or fictitious spaces for drugs.

Let $x_i(t)$ be the amount of the drug in the $i$ th compartment at time $t$.

We shall assume that the amount that can be transferred from the $i$ th to the $j$ th compartment $(j \neq i)$ in the time interval $(t, t + \Delta t)$ is $k_{ij}x_i(t)\Delta t + 0(\Delta t)$ where $k_{ij}$ is called the transfer coefficient from the $i$ th to the $j$ th compartment.

The total change $\Delta x_i$ in time $\Delta t$ is given by the amount entering the $i$ th compartment from other compartments which is reduced by the amount leaving the $i$ th compartment for other compartments including the zeroeth compartment that denotes the outside system.

Thus we get

$$\Delta x_i = -\sum_{\substack{j=0 \\ j \neq i}}^{n} k_{ij}x_i\Delta t + \sum_{\substack{j=1 \\ j \neq i}}^{n} k_{ji}x_j\Delta t + 0(\Delta t)$$

Dividing by $\Delta t$ and proceeding to the limit as $\Delta t \to 0$, we get

$$\begin{aligned}\frac{dx_i}{dt} &= -x_i \sum_{\substack{j=1\\ j\neq i}}^{n} k_{ij} + \sum_{\substack{j=1\\ j\neq l}}^{n} k_{ji}x_j\\ &= \sum_{j=1}^{n} k_{ji}x_j, \quad (i = 1, 2, \ldots, n,\end{aligned}$$

where we define

$$k_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{n} k_{ij}, \quad (i = 1, 2, \ldots, n)$$

In matrix notation, we have

$$\frac{dX}{dt} = KX \tag{2.3.21}$$

where

$$X(t) = \begin{bmatrix} x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}, \quad K = \begin{bmatrix} k_{11} & k_{21} & \ldots & k_{n1}\\ k_{12} & k_{22} & \ldots & k_{n2}\\ \ldots & \ldots & \ldots & \ldots\\ k_{1n} & k_{2n} & \ldots & k_{nn}\end{bmatrix} \tag{2.3.22}$$

If $X = Be^{\lambda t}$, when $B$ is a column matrix, (2.3.21) gives

$$\lambda Be^{\lambda t} = KBe^{\lambda t}$$

This gives a consistant system of equations to determine $B$ if

$$|K - \lambda I| = 0$$

where $I$ is $n \times n$ unit matrix. Thus $\lambda$ has to be an eigenvalue of the matrix $K$.

We note that all the diagonal elements of $K$ are negative, all the non-diagonal elements are non-negative and the sum of element of every column is greater than or equal to zero.

For such a matrix, it can be shown that the real parts of the eigenvalues are always less than or equal to zero, and the imaginary part is non-zero only when the real part is strictly less than zero.

Thus if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues then

$$\operatorname{Re}(\lambda_i) \leqslant 0$$
$$\operatorname{Im}(\lambda_i) \neq 0 \text{ only if } \operatorname{Rl}(\lambda_i) < 0$$

If the drug is injected at a constant rate given by the column vector $D$ with components $D_1, D_2, \ldots, D_n$, (2.3.21) becomes

$$dX/dt = KX + D \tag{2.3.23}$$

Equations (2.3.21) and (2.3.23) constitute the basic equations for the analysis of drug distribution in the $n$-compartment system. □

## Let us sum up:

- Compartment models utilising ODE systems.

## Check your progress:

1. Define Pharmokinetics.
2. Explain the Properties of the eigen values of the matrix.

## 2.4 Mathematical Modelling in Economics

Let $S(t), I(t), Y(t)$ be the Savings, Investment and National Income at time $t$, then it is assumed that

(i) Savings are proportional to national income, so that

$$S(t) = \alpha Y(t), \quad \alpha > 0$$

(ii) Investment is proportional to the rate of increase of national income so that

$$I(t) = \beta Y'(t), \quad \beta > 0$$

(iii) All savings are invested, so that

$$S(t) = I(t)$$

We get a system of three ordinary differential equations of first order for determining $S(t), Y(t), I(t)$. Solving we get

$$\begin{aligned}
S(t) &= I(t) \\
\beta Y'(t) &= \alpha Y(t) \\
\int \frac{dY}{Y} &= \frac{\alpha}{\beta} \int dt \\
\log Y &= \frac{\alpha}{\beta} t + \log A \\
\log \frac{Y}{A} &= \frac{\alpha}{\beta} t \\
\frac{Y}{A} &= e^{\frac{\alpha}{\beta} t} \\
Y &= A e^{\frac{\alpha}{\beta} t} \\
Y(t) &= Y_0 \exp(\frac{\alpha}{\beta} t) \quad \text{(By taking } Y(0) = Y_0 \text{ and solving)} \qquad (2.4.24)
\end{aligned}$$

(2.4.25)

and

$$I(t) = \alpha Y(0) e^{\alpha t/\beta} = S(t)$$

so that the national income, investment and savings all increase exponentially. □

**Problem 2.4.1.** *Formulate the Domar First Debt mathematical model in economics*

**Solution.**

Let $D(t), Y(t)$ denote the total national debt and total national income respectively, then we assume that (i) Rate at which national debt changes in proportional to national income so that

$$D'(t) = \alpha Y(t) \qquad (2.4.26)$$

(ii) National income increases at a constant rate, so that

$$Y'(t) = \beta \qquad (2.4.27)$$

Differentiating and simplifying we get

$$\begin{aligned} D''(t) &= \alpha Y'(t) \\ D''(t) &= \alpha\beta \end{aligned}$$

$$\text{Integrating, } D'(t) = \alpha\beta t + A \tag{2.4.28}$$

$$\text{Again integrating, } D(t) = \alpha\beta\frac{t^2}{2} + At + B \tag{2.4.29}$$

Since $D'(0) = \alpha Y(0)$, (2.4.28) becomes $A = \alpha Y(0)$

Also when t=0, (2.4.29) becomes $D(0) = B$

Again integrating (2.4.27), we get

$$Y(t) = \beta t + A$$

$$Y(0) = \beta t + Y(0)$$

So that

$$D(t) = D(0) + \alpha Y(0)t + \frac{1}{2}\alpha\beta t^2$$

$$Y(t) = Y(0) + \beta t$$

$$\frac{D(t)}{Y(t)} = \frac{D(0) + \alpha Y(0)t + \frac{1}{2}\alpha\beta t^2}{Y(0) + \beta t}$$

In this model, the ratio of national debt to national income tends to increase without limit.

□

**Problem 2.4.2.** *Formulate the Domar's Second Debt mathematical model in economics*

**Solution.**

In this model, the first assumption remains the same, but the second assumption is replaced by the assumption that the rate of increase of national income is proportional to the national income so that

$$Y'(t) = \beta Y(t) \tag{2.4.30}$$

Solving (2.4.26) and (2.4.30)

$$Y'(t) = \beta Y(t)$$

$$\begin{aligned}
\int \frac{dY}{Y} &= \beta \int dt \\
\log Y &= \beta t + \log A \\
\log \frac{Y}{A} &= \beta t \\
\frac{Y}{A} &= e^{\beta t} \\
Y &= Ae^{\beta t} \\
Y(0) &= A \quad \text{(By taking } t = 0 \text{ and solving)} \qquad (2.4.31)
\end{aligned}$$

$$\begin{aligned}
Y(t) &= Y(0)e^{\beta t} \\
\text{Then, } D'(t) &= \alpha Y(0)e^{\beta t} \\
\text{Integrating, } \ D(t) &= \frac{\alpha}{\beta} Y(0)e^{\beta t} + A \\
D(0) - \frac{\alpha}{\beta} Y(0) &= A \text{ (put t=0)} \\
\text{Hence, } D(t) &= \frac{\alpha}{\beta} Y(0)e^{\beta t} + D(0) - \frac{\alpha}{\beta} Y(0) \\
D(t) &= D(0) + \frac{\alpha}{\beta} Y(0) \left(e^{\beta t} - 1\right) \\
\frac{D(t)}{Y(t)} &= \frac{D(0)}{Y(0)e^{\beta t}} + \frac{\alpha}{\beta} \left(1 - e^{-\beta t}\right)
\end{aligned}$$

In this case $D(t)/Y(t) \to \alpha/\beta$ as $t \to \infty$. Thus when debt increases at a rate proportional to income, then if the ratio of debt to income is not to increase indefinitely, income must increase exponentially. □

**Problem 2.4.3.** *Derive the Allen's speculative mathematical model in economics*

**Solution.**

Let $d(t), s(t), p(t)$ denote the demand, supply and price of a commodity, then this model is given by

$$d(t) = \alpha_0 + \alpha_1 \beta(t) + \alpha_2 p'(t), \ \ \alpha_0 > 0, \alpha_1 < 0, \alpha_2 > 0 \qquad (2.4.32)$$

$$s(t) = \beta_0 + \beta_1 p(t) + \beta_2 p'(t), \ \ \beta_0 > 0, \beta_1 > 0, \beta_2 < 0 \qquad (2.4.33)$$

If $\alpha_2 = 0, \beta_2 = 0$ this gives Evan's price-adjustment model in which $\alpha_1 < 0$ since when price increasing, demand decreases and $\beta_1 > 0$ since when price increases, supply increases.

In Allen's model, coefficients $\alpha_2, \beta_2$ account for the effect of speculation.

If the price is increasing, demand increases in the expectation of the further increase in prices and supply decreases for the same reason.

For dynamic equilibrium

$$d(t) = s(t) \tag{2.4.34}$$

so that (2.4.32), (2.4.33) and (2.4.34) give

$$(\beta_2 - \alpha_2)\frac{dp}{dt} + (\beta_1 - \alpha_1)\,p(t) = \alpha_0 - \beta_0$$

$$\begin{aligned}\frac{dp}{dt} + \frac{(\beta_1 - \alpha_1)}{(\beta_2 - \alpha_2)}p(t) &= \frac{\alpha_0 - \beta_0}{(\beta_2 - \alpha_2)}\\ \frac{dp}{dt} - \lambda p(t) &= p_c \end{aligned} \tag{2.4.35}$$

where

$$p_c = \frac{\alpha_0 - \beta_0}{(\beta_2 - \alpha_2)}, \quad \lambda = \frac{\alpha_1 - \beta_1}{\beta_2 - \alpha_2}$$

Solving

$$\begin{aligned} p(t)e^{-\int \lambda dt} &= \int p_c e^{-\int \lambda dt} dt + C\\ p(t)e^{-\lambda t} &= -p_c\frac{e^{-\lambda t}}{\lambda} + C\\ p(t) &= \frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1)} + Ce^{\lambda t}\\ p(t) &= p_e + Ce^{\lambda t} \text{ (where, } p_e = \frac{\alpha_0 - \beta_0}{(\beta_1 - \alpha_1)}\text{)} && (2.4.36)\\ p(0) &= p_e + Ce^{\lambda t}\\ p(0) - p_e &= C \text{ (put t=0)} && (2.4.37) \end{aligned}$$

Hence (2.4.36) becomes

$$p(t) = p_e + (p(0) - p_e)\,e^{\lambda_t}$$

where

$$p_c = \frac{\alpha_0 - \beta_0}{(\beta_2 - \alpha_2)}, p_e = \frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1}, \quad \lambda = \frac{\alpha_1 - \beta_1}{\beta_2 - \alpha_2}$$

The behaviour of $p(t)$ depends on whether $p(\infty)$ or $p_e$ is large and whether $\lambda < 0$ or

$\lambda > 0$. The speculative model is highly unstable. □

**Problem 2.4.4.** *Derive the Samuelson's investment mathematical model in economics.*

**Solution.**

Let $K(t)$ represent the capital and $I(t)$ the investment at time $t$, then we assume that

(i) the investment gives the rate of increase of capital so that

$$\frac{dK}{dt} = I(t)$$

(ii) the deficiency of capital below a certain equilibrium level leads to an acceleration of the rate of investment proportional to this deficiency and a surplus of capital above this equilibrium level leads to a declaration of the rate of investment, again proportional to the surplus, so that

$$\frac{dI}{dt} = -m\left(K(t) - K_e\right)$$

where $K_e$ is the capital equilibrium level. If $k(t) = K(t) - K_e$, we get

$$\frac{dk}{dt} = I(t), \frac{dI}{dt} = -mk(t) \tag{2.4.38}$$

so that

$$-mk(t) = \frac{dI}{dt} = \frac{dI}{dt}\frac{dk}{dt} = I\frac{dI}{dk}$$

Integrating

$$\begin{aligned}
\int IdI &= -m\int kdk \\
\frac{I^2}{2} &= -m\frac{k^2}{2} + \frac{A^2}{2} \\
I^2 &= -mk^2 + A^2 \\
I(0)^2 &= -mk(0)^2 + A^2 \ (k_0 = k(0); I(0) = 0) \\
mk_0^2 &= A^2
\end{aligned}$$

Then

$$I^2 = m\left(k_0^2 - k^2\right); \quad k_0 = k(0); I(0) = 0$$

so that

$$\frac{dk}{dt} = -\sqrt{m}\sqrt{k_0^2 - k^2}$$

and

$$k(t) = k(0)\cos\sqrt{m}t$$
$$I(t) = -k(0)\sqrt{m}\sin\sqrt{m}t$$

so that both $k(t)$ and $I(t)$ oscillate with a time period $2\pi/\sqrt{m}$.

It will be noted that if we put $k(t) = x(t), I(t) = v(t)$, equation (2.4.38) are the equations for simple harmonic motion.

Thus the mathematical models for the oscillation of a particle in a simple harmonic motion and for the oscillation of capital about its equilibrium value are the same.

In this case, the rate of investment is slowed not only by excess capital as before, but it is also slowed by a high investment level so that (2.4.38) become

$$\frac{dk}{dt} = I(t), \quad \frac{dI}{dt} = -mk(t) - nI(t)$$

so that

or

$$I\frac{dI}{dk} + mk(t) + nI(t) = 0$$
$$\frac{d^2k}{dt^2} + n\frac{dk}{dt} + mk = 0$$

which are the equations for damped harmonic motion corresponding to the case when a particle performing SHM is acted as by a resistance force proportional to the velocity.

Let $p_r(t), s_r(t)$ and $d_r(t)$ be the price, supply and demand of a commodity in the $r$ th market, so that Evan's price adjustment model mechanism suggests

$$\frac{dp_r}{dt} = -\mu_r\left(s_r - d_r\right), \quad r = 1, 2, \ldots, n \tag{2.4.39}$$

Now we assume that the supply and demand of the commodity in the $r$ th market depends upon its price in all the markets, so that

$$s_r - d_r = c_r + \sum_{s=r}^{n} d_{rs}p_s \tag{2.4.40}$$

where $c_r$ 's and $d_{rs}$ 's are constants. From (2.4.39) and (2.4.40), we get

$$\frac{dp_r}{dt} = -\mu_r \left( c_r + \sum_{s=1}^{n} d_{rs} p_s \right), \quad r = 1, 2, \ldots, n$$

If $p_{1e}, p_{2e}, \ldots p_{ne}$ are the equilibrium prices in the $n$ markets and

$$P_{re} = p_r - p_{re},$$

we get

$$\frac{dP_r}{dt} = -\mu_r \sum_{s=1}^{n} d_{rs} P_s = \sum_{s=1}^{n} e_{rs} P_s, \quad r = 1, 2, \ldots, n$$

where

$$e_{rs} = -\mu_r d_{rs}$$

Substituting $P_r = A_r e^{\lambda t}$ and eliminating $A_1, A_2, \ldots, A_n$, we get

$$|\lambda I - E| = 0, \quad E = [e_{rs}]$$

Thus the equilibrium will be stable if all the eigen-value of the matrix $E$ have negative real parts.

If $d_{rs} = 0$ when $r \neq s$, the markets are independent so that non-zero value of some or all of these $d_{rs}'s$ introduce dependence among markets. □

**Problem 2.4.5.** *Explain Leontief's Open and Closed Dynamical Systems for Inter-industry Relation*

**Solution.**

We consider $n$ industries. Let

$x_{rs}$ = contribution from the $r$ th industry to the $s$ th industry per unit time

$x_r$ = contribution from the $r$ th industry to consumers per unit time

$X_r$ = total output of the $r$ th industry per unit time

$\xi_r$ = input of labour in the $r$ th industry

$p_r$ = price per unit of the product of the $r$ th industry

$w$ = wage per unit of labour per unit time

$Y$ = total labour input into the system

$S_{rs}$ = stock of the product of the $r$ th industry held by the $s$ th industry

$S_r$ = stock of the $r$ th industry.

Thus we get the following equations:

(i) From the principle of continuity, the rate of change of stock of the $r$ th industry = excess of the total output of the $r$ th industry per unit time over the contribution of the $r$ th industry to consumers and other industries per unit time, so that

$$\frac{d}{dt}S_r = X_r - x_r - \sum_{s=1}^{n} x_{rs}$$

and since

$$S_r = \sum_{s=1}^{n} S_{rs}$$

$$\frac{d}{dt}\sum_{s=1}^{n} S_{rs} = X_r - x_r - \sum_{s=1}^{n} x_{rs}, \quad (r = 1, 2, \ldots, n)$$

(ii) Since the total labour input into the system = sum of labour inputs into all industries, we get

$$Y = \sum_{r=1}^{n} \xi_r$$

(iii) Assuming the condition of perfect competition and no profit in each industry, we should have for each industry the value of input equal to the value of output so that

$$p_r X_r = \sum_{s=1}^{n} p_s x_{sr} + w\xi_r \quad (r = 1, 2, \ldots, n)$$

(iv) We further assume that the input coefficients

$$a_{rs} = \frac{x_{rs}}{X_s}, \quad b_{rs} = \frac{S_{rs}}{X_s}, b_r = \frac{\xi_r}{X_r}(r, s = 1, 2, \ldots, n)$$

are constants. We then get the equations

$$\frac{d}{dt}\sum_{s=1}^{n} b_{rs}X_s = X_r - x_r - \sum_{s=1}^{n} a_{rs}X_s, \quad (r = 1, 2, \ldots, n) \tag{2.4.41}$$

$$Y = \sum_{s=1}^{n} b_s X_s \tag{2.4.42}$$

$$p_r = \sum_{s=1}^{n} p_s a_{sr} + w b_r, \quad (r = 1, 2, \ldots, n) \tag{2.4.43}$$

We assume that the constants $a_{rs}, b_{rs}b_s$, are known.

We also assume that $x_1, x_2, \ldots, x_n$ and $w$ are given to us as function of time, then equations (2.4.41) determine $X_1, X_2, \ldots, X_n$ and then (2.4.42) determines $Y$ and finally (2.4.43) determine $p_1, p_2, \ldots, p_n$.

Thus if the final consumer's demands from all industries are known as functions of time, we can find the output which each industry must give and the total labour force required at any time.

Knowing the wage rate at any time, we can find the prices of products of different industries.

□

## Let us sum up:

Economical mathematical modelling based on ordinary differential equations of first order with some problems.

## Check your progress:

1. Discuss the types of mathematical models in economics.
2. What is the difference between Domar Macro Model, Domar First Debt Model and Domar's Second Debt Model?

## 2.5 Mathematical Models in the Medicine and Arms Race Battles

Let $x(t), y(t)$ be the blood sugar and insulin levels in the blood stream at time $t$.

The rate of change $dy/dt$ of insulin level is proportional to (i) the excess $x(t) - x_0$ of sugar in blood over its fasting level, since this excess makes the pancreas secrete insulin into the blood stream (ii) the amount $y(t)$ of insulin since insulin left to itself tends to decay at a rate proportional to its amount and (iii) the insulin dose $d(t)$ injected per unit time.

This gives

$$\frac{dy}{dt} = a_1 (x - x_0) H (x - x_0) - a_2 y + a_3 d(t) \tag{2.5.44}$$

where $a_1, a_2, a_3$ are positive constants and $H(x)$ is a step function which takes the value unity when $x > 0$ and taken the value zero otherwise.

This occurs in (95) because if blood sugar level is less than $x_0$, there is no secretion of insulin from the pancreas.

Again the rate of change $dx/dt$ of sugar level is proportional to (i) the product $xy$ since the higher the levels of sugar and insulin, the higher is the metabolism of sugar (ii) $x_0 - x$ since if sugar level falls below fasting level, sugar is released from the level stores to raise the sugar level to normal (iii) $x - x_0$ since if $x > x_0$, there is a natural decay in sugar level proportional to its excess over fasting level (iv) function of $t - t_0$ where $t_0$ is the time at which food is taken

$$\begin{aligned} \frac{dx}{dt} &= -b_1 xy + b_2 (x_0 - x) H (x_0 - x) - b_3 (x - x_0) H (x - x_0) \\ &\quad + b_4 z (t - t_0) \end{aligned} \tag{2.5.45}$$

where a suitable form for $z (t - t_0)$ can be

$$\begin{aligned} z (t - t_0) &= 0, \quad t < t_0 \\ &= Q e^{-\alpha (t - t_0)}, \quad t > t_0 \end{aligned}$$

Equations (2.5.44) and (2.5.45) give two simultaneous differential equations to determine $x(t)$ and $y(t)$. These equation can be numerically integrated.

Let $x(t), y(t)$ be the expenditures on arms by two countries $A$ and $B$, then the rate of change $dx/dt$ of the expenditure by the country $A$ has a term proportional to $y$, since the larger the expenditure in arms by $B$, the larger will be the rate of expenditure on arms by $A$.

Similarly it has a term proportional to $(-x)$ since its own arms expenditure has an inhibiting effect on the rate of expenditure on arms by $A$.

It may also contain a term independent of the expenditures depending on mutual suspicions or mutual goodwill.

With these considerations, Richardson gave the model

$$\frac{dx}{dt} = ay - mx + r, \frac{dy}{dt} = bx - ny + s \tag{2.5.46}$$

Here $a, b, m, n$ are all $> 0. r$ and $s$ will be positive in the case of mutual suspicions and negative in the case of mutual goodwill.

A position of equilibrium $x_0, y_0$, if it exists, will be given by

$$\begin{aligned} mx_0 - ay_0 - r &= 0 \\ bx_0 - ny_0 + s &= 0 \end{aligned} \quad \text{or} \quad \frac{x_0}{-as - nr} = \frac{y_0}{-br - ms} = \frac{1}{-mn + ab}$$

$$x_0 = \frac{as + nr}{mn - ab}, y_0 = \frac{ms + br}{mn - ab}.$$

If $r, s$ are positive, a position of equilibrium exists if $ab < mn$. If $X = x - x_0, Y = y - y_0$, we get

$$\frac{dX}{dt} = aY - mX, \frac{dY}{dt} = bX - nY$$

$X = Ae^{\lambda t}, Y = Be^{\lambda t}$ will satisfy these equations if

$$\begin{vmatrix} \lambda + m & -a \\ -b & \lambda + n \end{vmatrix} = 0, \quad \lambda^2 + \lambda(m + n) + mn - ab = 0 \tag{2.5.47}$$

Now the following cases arise:

(i) $mn - ab > 0, r > 0, s > 0$. In this case $x_0 > 0, y_0 > 0$ and from (2.5.47) $\lambda_1 < 0, \lambda_2 < 0$. As such there is a position of equilibrium and it is stable.

(ii) $mn - ab > 0, r < 0, s < 0$, there is no position of equilibrium since $x_0 < 0, y_0 < 0$. However since $\lambda_1 < 0, \lambda_2 < 0, X(t) \to 0, Y(t) \to 0$ as $t \to \infty$, so that $x(t) \to x_0, y(t) \to y_0$. However $x_0$ and $y_0$ are negative and populations cannot become negative. In any case to become negative, they have to pass through zero values. As such, as $x(t)$ becomes zero, (2.5.46) is modified to

$$\frac{dy}{dt} = -ny + s$$

and since $s < 0, y(t)$ decreases till it reaches zero. Similarly if $y(t)$ becomes zero first, (2.5.46) is modified to

$$\frac{dx}{dt} = -mx + r$$

and since $r < 0, x(t)$ decreases till it reaches zero. Thus if $mn - ab > 0$, $r < 0, s < 0$, there will ultimately be complete disarmament.

(iii) $ma - ab < 0, r > 0, s > 0$. These give $x_0 < 0, y_0 < 0$, one of $\lambda_1, \lambda_2$ is positive and the other is negative. In this case there will be a runaway arms race.

(iv) $ma - ab < 0, r < 0, s < 0$. These give $x_0 > 0, y_0 > 0$ one of $\lambda_1, \lambda_2$ is positive and the other is negative. In this case there will be a runaway arms race or disarmament depending on the initial expenditure on arms.

Let $x(t)$ and $y(t)$ be the strengths of the two forces engaged in combat and let $M$ and $N$ be the fighting powers of individuals depending on physical fitness, types of arms and training, then Lanchester postulated that the reduction in strength of each force is proportional to the effective fighting strength of the opposite force, so that

$$\frac{dx}{dt} = -ayN, \quad \frac{dy}{dt} = -axM$$

giving

$$\frac{dx}{yN} = \frac{dy}{xM} \quad \text{or} \quad Mx^2 - Ny^2 = \text{constant}$$

If the proportional reduction of strengths in the two forces are the same

$$\frac{1}{x}\frac{dx}{dt} = \frac{1}{y}\frac{dy}{dt} \quad \text{or} \quad \frac{Ny}{x} = \frac{Mx}{y} \text{ or } Mx^2 = Ny^2$$

This is the square law. The fighting strength of an army depends on the square of its numerical strength and directly on the fighting quality of individuals. □

**Problem 2.5.1.** *Write International Trade Model.*

**Solution.** Since international trade is beneficial to all parties, we can consider the model

$$\frac{dx_1}{dt} = a_{12}x_1x_2 + a_{13}x_1x_3 + \ldots + a_{1n}x_1x_n$$
$$\frac{dx_2}{dt} = a_{21}x_2x_1 + a_{23}x_2x_3 + \ldots + a_{2n}x_2x_n$$
$$\ldots\ldots\ldots\ldots\ldots..$$
$$\frac{dx_n}{dt} = a_{n1}x_nx_1 + a_{n2}x_nx_2 + \ldots + a_{nn-1}x_nx_{n-1}$$

where all $a_{ij}$ 's are positive. An equilibrium position is $(0, 0, \ldots, 0)$ and this is stable. □

## Let us sum up:

- Medicine, arms race battles.
- Blood sugar insulin level
- International trade model.

## Check your progress:

1. What are the types of Mathematical models in the medicine , Arm race battles and international trade in terms of systems of ODEs?
2. What is the rate of change $dy/dt$ of insulin?

## 2.6 Mathematical Modelling in Dynamics

If a particle moves in two dimensional space, we want to determine $x(t)$, $y(t)$, its coordinates at any time $t$ and $u(t), v(t)$ its velocity components at the same time. Similarly for the motion of a particle in three dimensions, we have to determine $x(t), y(t), z(t), u(t), v(t), w(t)$. For motion of a rigid body in three dimensional space, we require twelve quantities at time $t$ viz. six coordinates and velocities of its centre of gravity and six angles and angular velocities about the centre of gravity.

Since equation of motion are based on the principle: mass $\times$ acceleration in any direction = force in that direction, we get systems of second order differential equations. However since acceleration is the rate of change of velocity and velocity is the rate of change of displacement, we can decompose one ordinary differential equation of the second order into two ordinary differential equations of the first order.

We discuss below the motion of a particle in a plane under gravity. More general dynamical motions will be discussed in the next chapter. □

### 2.6.1 Motion of a Projectile

A particle of mass $m$ is projected from the origin in vacuum with velocity $V$ inclined at an angle $\alpha$ to the horizontal. Suppose at time $t$, it is at position $x(t), y(t)$ and its horizontal and vertical velocity components are $u(t), v(t)$ respectively, then the equations of motion are:

$$m\frac{du}{dt} = 0 \quad m\frac{dv}{dt} = -mg$$

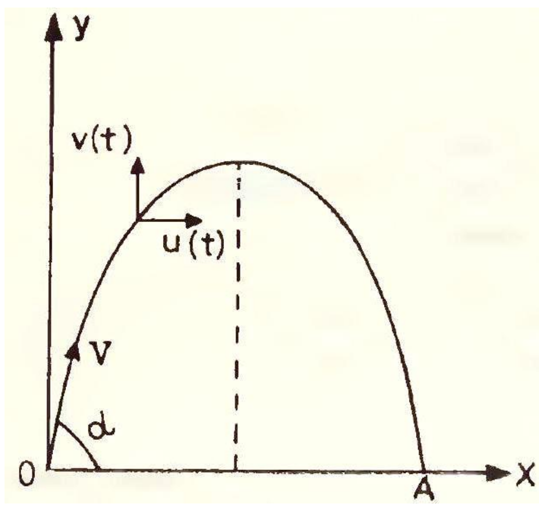


Figure 2.3

Integrating

$$\int du = \int 0dt, \int dv = \int gdt$$

$$u = A, v = gt + B$$

$$V\cos\alpha = A, V\sin\alpha = B \quad \text{when t=0, u(0)=Vcos}\,\alpha,\ \text{v(0)} = \text{Vsin}\,\alpha$$

$$u = V\cos\alpha, \quad v = V\sin\alpha - gt,$$

$$\frac{dx}{dt} = V\cos\alpha, \frac{dy}{dt} = V\cdot\sin\alpha - gt$$

Integrating again

$$x = V\cos\alpha t, \quad y = V\sin\alpha t - \frac{1}{2}gt^2 \tag{2.6.48}$$

Eliminating $t$ between these two equations, we get from (2.6.48) that

$$t = \frac{x}{\cos\alpha}$$
$$\text{Then, } y = \frac{xV\sin\alpha}{V\cos\alpha} - \frac{1}{2}\frac{gx^2}{V^2\cos^2\alpha}$$
$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{V^2\cos^2\alpha}$$

which is a parabola, since the terms of the second degree form a perfect square. The parabola cuts $y = 0$, when

$$x = 0 \quad \text{or} \quad x = \frac{V^2\sin 2\alpha}{g}$$

corresponding to position 0 and $A$ in Figure 2.3 so that the range of the particle is given by

$$R = \frac{V^2\sin 2\alpha}{g}$$

Putting $y = 0$ in (2.6.48) we get

$$t = 0 \quad \text{or} \quad t = \frac{2V\sin\alpha}{g}$$

This gives the time $T$ of flight. Since the horizontal velocity is constant and equal to $V\cos\alpha$, the total horizontal distance travelled is

$$V\cos\alpha(2V\sin\alpha)/(g) = V^2\sin 2\alpha/g$$

which gives us the same range. □

### 2.6.2 External Ballistics of Gun Shells

To study the motion of gun shells, the following additional factors have to be taken into account:

(i) air resistance which may be proportional to $v^n$, but the power $n$ can be different for different ranges of $v$

(ii) wind velocity, humidity and pressure

(iii) rotation of the earth

(iv) the fact that shell is a rigid body and as such both motion of its centre of gravity and motion about the centre of gravity have to be studied. When the shell comes out of the gun, it is rotating with a large angular velocity.

It is obvious that the problems will be quite complex, but all these problems have been solved and powerful computers have been developed to solve these problems because of their importance to defence.

In the case of intercontinental ballistic missiles, heating and aerodynamic effects have also to be considered.

## Let us sum up:

- Modelling in dynamics.
- Motion of a projectile.
- External ballistics of gun shells.

## Check your progress:

1. What is the range of the particles projected from the origin in vacuum with velocity $V$ inclined at an angle $\alpha$ to the horizontal?
2. What are the additional factors that are used to study External Ballistics of Gun Shells?

## Summary:

In this unit, we have developed a model for analysis of the population dynamics. In addition, studied the spread of diseases through epidemic models and infectious diseases.

Further, we discussed the compartment model, economics, medicine and the arms race. Finally, we solved simple problems in first-order ordinary differential equations.

## Glossary:

Prey-predator model, Competition models, Multi species model, Epidemic model.

## Self assessment questions

1. What are the equilibrium points of the compartmental models and multispecies models ?
2. What is the difference between Samuelson's Investment Modeland Samuelson's Modified Investment Model ?
3. Obtain the steady-state solution of Leontief?s model.
4. Show that for the Lanchester model, the trajectories are hyperbolas, all of which have the same asymptotes.
5. Show that both the range and maximum height of a projectile are reduced by air resistance.

## Exercise

1. Draw some trajectories for the model $\frac{dx}{dt} = x(1 - 0.1y)$ , $\frac{dy}{dt} = -y(1 - 0.1x)$.
2. Discuss the modification of the prey-predator model when
    - the predator population is harvested at a constant rate $h_1$ or
    - the prey population is harvested at a constant rate $h_2$ or
    - both species are harvested at constant rates.
3. Solve $SIS$ model when $\beta$ is a known function of $t$ .

4. Let dose $D$ be given at time $0, T, 2T, 3T, \cdots$, Find $X(nT-0), X(nT+1), X(nT+t)$, $(0 < t < T)$

5. Discuss the solution of Allen?s speculative model when (i) $\lambda > 0$ (ii) $\lambda < 0$ (iii)$p_e > p(0)$(iv)$p_e < p(0)$ and interpret the solution in each case

6. For the model $\frac{dN_1}{dt} = N_1(a_1 - b_1N_1 - b_2N_2)$, $\frac{dN_2}{dt} = N_2(a_2 - c_1N_1 - c_2N_n)$, $a_1, a_2 > 0$,$b_1, b_2 > 0$,$C_l, C_2 > 0$. find the positions of equilibrium and discuss their stability. Draw also the direction fields and possible trajectories.

## Answers for check your progress

### section 2.1

1. (i) If there are no predators, the prey species will grow at a rate proportional to the population of the prey species,(ii) if there are no prey, the predator species will decline at a rate proportional to the population of the predator species, (iii) the presence of both predators and preys is beneficial to growth of predator species and is harmful to growth of prey species. More specifically the predator species increases and the prey species decreases at rates proportional to the product of the two populations.

2. The intersection will imply two different slopes at the same point.

3. Two equilibrium positions are $(0, 0)$ and $(p/q, a/b)$.

4. This method will enable us to discuss only local stability of a position of equilibrium, i.e. this will decide that if the populations of different species are changed slightly from these equilibrium values, whether the population sizes will return to their original equilibrium values or not. The problem of discussing the global stability i.e. of discussing whether the populations will return to these equilibrium values, whatever be the magnitudes of the disturbances, is a more difficult problem and it is possible to solve this problem in special cases only.

## Section 2.2

1. A simple epidemic model, A Susceptible-Infected-Susceptible (SIS) Model, SIS Model with Constant Number of Carriers, Simple Epidence Model with Carriers, Model with Removal, Model with Removal and Immigration.

2. In simple epidence model, only carriers spread the disease and their number decreases exponentially with time as these are identified and eliminated and in model with removal infected persons are removed by death or hospitalisation at a rate proportional to the number of infectives.

## Section 2.3

1. Pharmokinetics (also called drug kinetics or tracer kinetics or multi-compartment analysis) deals with the distribution of drugs, chemicals,tracers or radio-active substances among various compartments of the body where compartments are real or fictitious spaces for drugs.

2. The real parts of the eigenvalues are always less than or equal to zero, and the imaginary part is non-zero only when the real part is strictly less than zero.

## Section 2.4

1. Domar Macro Model, Domar First Debt Model, Domar's Second Debt Model, Allen's Speculative Model, Samuelson's Investment Model, Samuelson's Modified Investment Model, Stability of Market Equilibrium, Leontief's Open and Closed Dynamical Systems for Inter-industry Relation.

2. In Domar Macro Model, the national income, investment and savings all increase exponentially. In Domar's First Debut Model ,the ratio of national debt to national income tends to increase without limit. In the Domar's Second Debut Model, when debt increases at a rate proportional to income, then if the ratio of debt to income is not to increase indefinitely, income must increase exponentially.

### Section 2.5

1. A Model for Diabetes Mellitus, Richardson's Model for Arms Race, Lanchester's Combat Model,International Trade Model

2. The rate of change of insulin is proportional to the excess of sugar in blood over its fasting level, since this excess makes the pancreas secrete insulin into the blood stream (ii) the amount $y(t)$ of insulin since insulin left to itself tends to decay at a rate proportional to its amount and (iii) the insulin dose $d(t)$ injected per unit time.

### Section 2.6

1. $R = \frac{V^2 \sin 2\alpha}{g}$.

2. (i) air resistance which may be proportional to $\nu^n$, but the power n can be different for different ranges of $\nu$(ii) wind velocity, humidity and pressure (iii) rotation of the earth etc.,

## References:


1. J.N. Kapur, Mathematical Modelling, Wiley Eastern Limited, New Delhi, 4th Reprint, May 1994.


## Suggested Reading:


1. M. Braun, C.S. Coleman and D. A. Drew, Differential Equation Models, 1994.

2. A.C. Fowler, Mathematical Models in Applied Sciences, Cambridge University Press, 1997.

3. Walter J. Meyer, Concepts of Mathematical Modeling. Courier Corporation, 2012.

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# UNIT - 3

# Unit 3

# Mathematical Modelling Through Ordinary Differential Equations of Second Order

## Objectives:

- Recall the planetary motions, Circular Motion and Motion of Satellites
- Know to how make model planetary motions through linear differential equations of second order.
- To solve simple problems in second-order ordinary differential equations.

## 3.1 Mathematical Modelling of Planetary Motions

### 3.1.1 Need for the Study of Motion Under Central Forces

Every planet moves mainly under the gravitational attractive force exerted by the Sun.

If $S$ and $p$ are masses of the Sun and the planet and $G$ is the universal constant of gravitation, then the forces of gravitational attraction on the Sun and planet are both $GSP/r^2$, where $r$ is the distance between the Sun and the planet.

Accordingly the acceleration (3.1) of the Sun towards the planet is $GP/r^2$ and the acceleration of the planet towards the Sun is $GS/r^2$.

The acceleration of the planet relative to the Sun is

$$G(S+P)/r^2 = \mu/r^2.$$

Now we take the Sun as fixed, then the planet can be said to move under a central force $\mu/r^2$ per unit mass i.e. under a force which is always directed towards a fixed centre $S$.

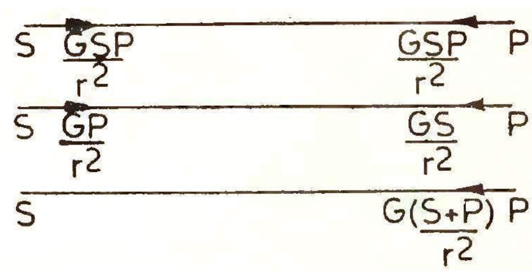



Figure 3.1

We shall for the present also regard $P$ as a particle so that to study the motion of the planet, we have to study the motion of a particle moving under a central force.

We can take $S$ as origin so that the central force is always along the radius vector.

To study this motion, it is convenient to use polar coordinates and to find the components of the velocity and acceleration along and perpendicular to the radius vector.

□

### 3.1.2 Components of Velocity and Acceleration Vectors along Radial and Transverse Directions

As the particle moves from $P$ to $Q$, the displacement along the radius vector

$$= ON - OP = (r+\Delta r)\cos\Delta\theta - r$$

and the radial component $u$ of velocity is

$$u = \underset{\Delta t\to 0}{\mathrm{Lt}}(r+\Delta r)\frac{\cos\Delta\theta - r}{\Delta t}$$
$$= \mathrm{Lt}_{\Delta t\to 0}\frac{\Delta r}{\Delta t} = \frac{dr}{dt}$$

(Since, $\Delta\theta = 0, cos\Delta\theta = 1$)

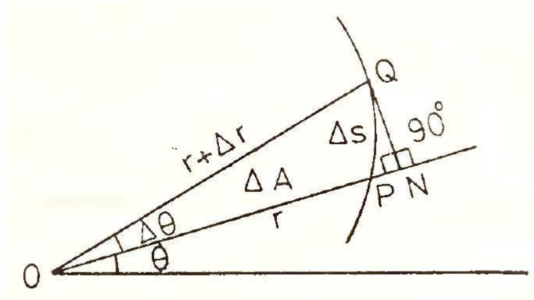



Figure 3.2

Similarly the displacement perpendicular to the radius vector

$$= (r + \Delta r) \sin \Delta\theta$$

and the transverse component $v$ of the velocity is given by

$$\begin{aligned} v &= \text{Lt}_{\Delta t \to 0} \frac{(r + \Delta r) \sin \Delta\theta}{\Delta t} \\ &= \text{Lt}_{\Delta t \to 0} r \frac{\sin \Delta\theta}{\Delta\theta} \frac{\Delta\theta}{\Delta t} \quad (when\ \Delta t \to 0,\ \sin \Delta\theta \to 0) \\ &= r \frac{d\theta}{dt} \end{aligned}$$

As such the velocity components in polar coordinates are

$$u = \frac{dr}{dt} = r' \quad \text{and} \quad v = r\frac{d\theta}{dt} = r\theta'$$

Now the change in the velocity along the radius vector

$$= (u + \Delta u) \cos \Delta\theta - (v + \Delta v) \sin \Delta\theta - u$$

and the radial component of acceleration

$$\begin{aligned} &= \text{Lt}_{\Delta t \to 0} \frac{(u + \Delta u) \cos \Delta\theta - (v + \Delta v) \sin \Delta\theta - u}{\Delta t} \\ &= \text{Lt}_{\Delta t \to 0} \frac{\Delta u - v\Delta\theta}{\Delta t} \quad (when\ \Delta t \to 0,\ \sin \Delta\theta \to 0\ and\ \cos \Delta\theta = 1) \\ &= \frac{du}{dt} - v\frac{d\theta}{dt} \\ &= \frac{d}{dt}(r') - r\theta'\theta' \\ &= r'' - r\theta'^2 \end{aligned}$$

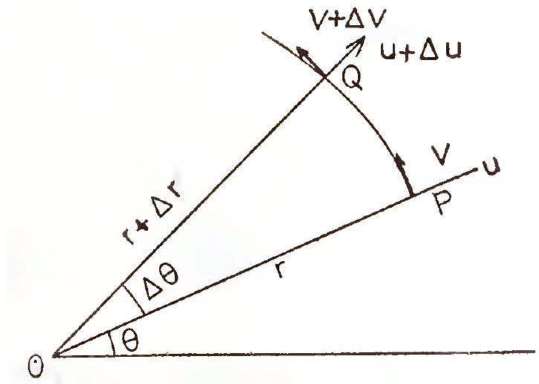



Figure 3.3

Similarly the transverse component of acceleration

$$
\begin{aligned}
&= \text{Lt}_{\Delta t \to 0} \frac{(u+\Delta u)\sin\Delta\theta + (v+\Delta v)\cos\Delta\theta - v}{\Delta t} \\
&= \text{Lt}_{\Delta t \to 0} \frac{u\Delta\theta + \Delta v}{\Delta t} \;\; (when\ \Delta t \to 0,\ \sin\Delta\theta \to 0\ and\ \cos\Delta\theta = 1) \\
&= u\frac{d\theta}{dt} + \frac{dv}{dt} \\
&= r'\theta' + \frac{d}{dt}(r\theta') \\
&= \frac{1}{r}\frac{d}{dt}\left(r^2\theta'\right) \;\; (Since\ \frac{1}{r}\frac{d}{dt}(r^2\theta') = (\frac{dr}{dt}\theta') + \frac{d}{dt}(r\theta'))
\end{aligned}
$$

Thus the radial and transverse components of acceleration are

$$r'' - r\theta'^2 \text{ and } \frac{1}{r}\frac{d}{dt}\left(r^2\theta'\right)$$

□

### 3.1.3 Motion Under a Central Force

Let the force acting on a particle of mass $m$ be $mF(r)$ and let it be directed towards the origin, then the equations of motion are

$$m\left(r'' - r\theta'^2\right) = -mF(r) \tag{3.1.1}$$

$$\frac{m}{r}\frac{d}{dt}\left(r^2\theta'\right) = 0 \tag{3.1.2}$$

From (3.1.2), Integrating $\int \frac{m}{r}\frac{d}{dt}(r^2\theta') = \int 0 \; dt$

$$r^2\theta' = \text{ constant } = h(\text{ say }), \tag{3.1.3}$$

then (3.1.1) gives

$$r'' - r\theta'^2 = -F(r) \tag{3.1.4}$$

We can eliminate $t$ between (3.1.3) and (3.1.4) to get a differential equation between $r$ and $\theta$. We find it convenient to use $u = 1/r$ instead of $r$, so that making use of (3.1.3), we get

$$r' = \frac{dr}{dt} = \frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{h}{r^2} = -h\frac{du}{d\theta}$$

and

$$\begin{aligned} r'' = \frac{d}{dt}\left(-h\frac{du}{dt}\right) &= \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\frac{d\theta}{dt} \\ &= -h\frac{d^2u}{d\theta^2}hu^2 = -h^2u^2\frac{d^2u}{d\theta^2} \end{aligned} \tag{3.1.5}$$

From (3.1.3), (3.1.4) and (3.1.5)

$$-F(r) = -h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4 = -h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

or

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2} \tag{3.1.6}$$

where $F$ can be easily expressed as a function of $u$.

This is the differential equation of the second order whose integration will give the relation between $u$ and $\theta$ or between $r$ and $\theta$ i.e. the equation of the path described by a particle moving under a central force $F$ per unit mass. □

### 3.1.4 Motion Under the Inverse Square Law

If the central force per unit mass is $\mu/r^2$ or $\mu u^2$, Equation (3.1.6) gives

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}$$

Integrating this linear equation with constant coefficients, we get

$$\begin{aligned} u &= A\cos(\theta-\alpha) + \frac{\mu}{h^2} \\ \text{or } \frac{h^2/u}{r} &= \frac{L}{r} = 1 + e\cos(\theta-\alpha); h^2 = \mu L, \end{aligned} \tag{3.1.7}$$

which represents a conic with a focus at the centre of force. Thus if a particle moves under a central force $\mu/r^2$ per unit mass, the path is a conic section with a focus at the centre. The conic can be an ellipse, parabola, or hyperbola according as $e \lesseqgtr 1$.

Now the velocity $V$ of the particle is given by

$$\begin{aligned} V^2 = r'^2 + r^2\theta'^2 &= \left(\frac{dr}{du}\frac{du}{d\theta}\frac{d\theta}{dt}\right)^2 + \frac{1}{u^2}\left(hu^2\right)^2 \\ &= h^2\left(\frac{du}{d\theta}\right)^2 + h^2u^2 \end{aligned} \tag{3.1.8}$$

Using (3.1.7)

$$L\frac{du}{d\theta} = -e\sin(\theta-\alpha) \tag{3.1.9}$$

From (3.1.8) and (3.1.9)

$$\begin{aligned} V^2 &= \mu L\left(\frac{e^2\sin^2(\theta-\alpha)}{L^2} + \frac{(1+e\cos(\theta-\alpha)^2}{L^2}\right) \\ &= \frac{\mu}{L}\left(1 + e^2 + 2e\cos(\theta-\alpha)\right) \\ &= \frac{\mu}{L}\left(e^2 - 1 + 2(1 + e\cos(\theta-\alpha))\right) \\ &= \frac{\mu}{L}\left(e^2-1\right) + \frac{2\mu}{r} \end{aligned}$$

If the path is an ellipse $\quad L = a\,(1-e^2)$

If the path is a parabola $e = 1$

If the path is a hyperbola $L = a\,(e^2-1)$, so that

$$
\begin{aligned}
V^2 &= \mu\left(\frac{2}{r}+\frac{1}{a}\right) \text{ in the case of a hyperbola} \\
&= \mu\left(\frac{2}{r}\right) \text{ in the case of a parabola} \\
&= \mu\left(\frac{2}{r}-\frac{1}{a}\right) \text{ in the case of an ellipse.}
\end{aligned}
$$

Thus if the particle is projected with velocity $V$ from a point at a distance $r$ from the centre of force, the path will be a hyperbola, parabola or ellipse according as

$$V^2 - \frac{2\mu}{r} \gtreqless 0$$

We have proved that if the central force is $\mu/r^2$ per unit mass, the path is a conic section with the centre of forces at one focus. Conversely if we know that the path is a conic section

$$\frac{L}{r} = Lu = 1 + e\cos(\theta - \alpha),$$

with a focus at the centre of force, then the force per unit mass is given by

$$
\begin{aligned}
F &= h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) \\
&= h^2u^2\left(\frac{-e\cos(\theta-\alpha)}{L}+\frac{1+\cos(\theta-\alpha)}{L}\right) \\
&= \frac{h^2}{L}u^2 = \frac{\mu}{r^2}
\end{aligned}
$$

so that the central force follows the inverse square law.

Since all planets are observed to move in elliptic orbits with the Sun at one focus, it follows that the law of attraction between different planets and Sun must be the inverse square law. □

### 3.1.5 Kepler's Laws of Planetory Motions

On the basis of the long period of observations of planetory motions by his predecessors and by Kepler himself, Kepler deduced the following three laws of motion empirically

(i) Every planet describes an ellipse with the Sun at one focus

(ii) The radius vector from the Sun to a planet describes equal areas in equal intervals of time.

(iii) The squares of periodic time of planets are proportional to the cubes of the semimajor axes of the orbits of the planets

We can deduce all these three laws from the mathematical modelling of planetory motion discussed above, when the law of attraction is the inverse square law.

(i) We have already seen that under the inverse square law, the path has to be a conic section and this includes elliptic orbits.

(ii) Since $r^2\theta' = h$, we get

$$\mathrm{Lt}_{\Delta t \to 0} \frac{1}{2}\frac{r^2 \Delta\theta}{\Delta t} = \frac{1}{2}h \tag{3.1.10}$$

From Figure 3.2, the area $\triangle A$ bounded by radius vectors $OP$ and $OQ$ and the arc $PQ$ is $1/2r^2 \sin\Delta\theta$ so that (3.1.10) gives

$$\frac{dA}{dt} = \frac{1}{2}h$$

and the rate of description of sectorical area is constant and equal areas are described in equal intervals of time. This is Kepler's second law.

(iii) The total area of the ellipse is $\pi ab$ and since the areal velocity is $\frac{1}{2}h$, the periodic time $T$ is given by

$$T = \frac{\pi ab}{\frac{1}{2}h} = \frac{2\pi ab}{\sqrt{\mu L}} = \frac{2\pi ab}{\sqrt{\mu}\sqrt{b^2/a}} = \frac{2\pi}{\sqrt{\mu}}a^{3/2}$$

For two different planets of masses $P_1, P_2$, and semiaxes of orbits $a_1, a_2$, this gives or

$$\frac{T_1}{T_2} = \frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\frac{a_1^{3/2}}{a_2^{3/2}} = \frac{\sqrt{G\,(S+P_2)}}{\sqrt{G\,(S+P_1)}}\frac{a_1^{3/2}}{a_2^{3/2}}$$
$$\frac{T_1^2}{T_2^2} = \frac{S+P_2}{S+P_1}\frac{a_1^3}{a_2^3} = \frac{1+\frac{P_2}{S}}{1+\frac{P_1}{S}}\frac{a_1^3}{a_2^3}$$

Since $P_1, P_2$ are very small compared with $S$, this gives, as a very good approximation

$$\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}$$

which is Kepler's third law of planetory motion.

Deduction of Kepler's three laws of planetory motion from the universal law of gravitation was an important success of mathematical modelling. Results which took hundreds of years to obtain by observation could be obtained in a very short time by using mathematical modelling. □

**Remark 3.1.1.** *Here we have neglected the forces of attraction of other planets on the given planet. These are very small as compared with the attractive force of the Sun. However these can be taken into account. In fact possibly the most sensational achievement of mathematical modelling was achieved when the discrepancies from the above theory observed in the motion of planets were explained as possibly due to the existence of another small planet. The position of this planet, not observed till that time, was calculated, and when the telescope was pointed out to that position in the sky, the planet was there!*

*Again the occurrence of many of the fundamental particles in physics has been theoretically predicted on the basis of mathematical modelling.*

*The advantages of developing a successful theoretical model over relying in purely observational and empirical models are that (i) this development can suggest development of mathematical models for similar situations elsewhere and those new models can later be validated and (ii) the theoretical models, unlike empirical models, can be generalised. Thus the model developed by Newton for planetory motion could be easily extended to apply to motion of artificial satellites. Similarly in urban transportation, a gravity model was developed by trial and error and ad hoc empirical methods extending over a period of thirty to forty years. When the same model was obtained theoretically from the principle of maximum entropy, it could be easily generalised for many more complex situations than could ever be handled by the empirical methods.*

## Let us sum up:

- Importance of researching motion under central forces.
- Elements of the acceleration and velocity vectors in the radial and transverse directions.

- Motion under a central foece.
- Motion under the inverse square law.
- The laws of Kepler for planetary motion.

## Check your progress:

1. Find the e radial and transverse components of acceleration.
2. What are the three laws of Kepler's Laws of Planetory Motions ?

## 3.2 Mathematical Modelling of Circular Motion and Motion of Satellites

### 3.2.1 Circular Motion

When a particle moves in a circle of radius $a$ so that $r = a$,

the radial component of velocity $= r' = 0$,

the transverse component of velocity $= r\theta' = a\theta'$

the radial component of acceleration $= r'' - r\theta'^2 = -a\theta'^2$,

the transverse component of acceleration $= \frac{1}{r}\frac{d}{dt}\left(r^2\theta'\right) = \frac{1}{a}\frac{d}{dt}\left(a^2\theta'\right) = a\theta''$.

Thus the velocity is $a\theta'$ along the tangent and the acceleration has two components $a\theta''$ along the tangent and $a\theta'^2$ along the normal.

If a particle moves in a circle of radius $a$, its equations of motion are $ma\theta''$ = external force in the direction of the tangent $ma\theta'^2$ = external force in the direction of the inward normal.

Thus if a particle is attached to one end of a string, the other end of which is fixed and the particle moves in a vertical circle, the equations of motion are (Figure 3.4)

$$ma\theta'' = -mg\sin\theta \tag{3.2.11}$$

$$ma\theta'^2 = T - mg\sin\theta \tag{3.2.12}$$

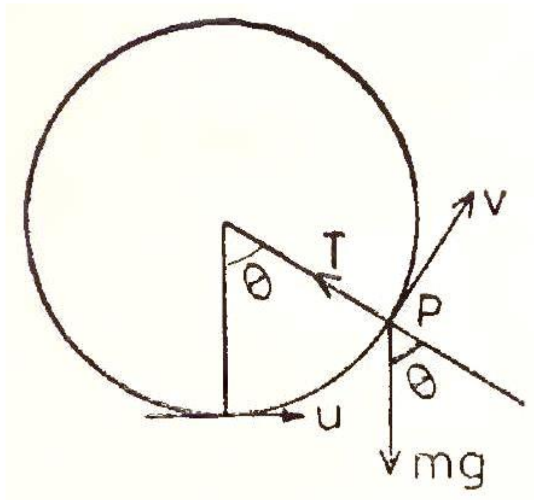



Figure 3.4

If $\theta$ is small, (3.2.11) gives

$$\theta'' = -\frac{g}{a}\theta \quad (|sin\theta| = 1)$$

which is the equation for a simple harmonic motion. Thus for small oscillations of a simple pendulum, the time period is

$$T = 2\pi\sqrt{a/g}$$

If $\theta$ is not necessarily small, integration of (3.2.14) gives

$$\begin{aligned} a\int \theta'' d\theta &= g\int \sin\theta \\ a\theta' &= -g\cos\theta + A \end{aligned}$$

$$a\theta'^2 = 2g\cos\theta + \text{ constant}$$

If the particle is projected from the lowest point with velocity $u$, then $a\theta' = u$ when $\theta = 0$, so that

$$a\theta'^2 = \frac{v^2}{a} = \frac{u^2}{a} - 2g(1 - \cos\theta)$$

where $v$ is the velocity of the particle, so that

$$v^2 = u^2 - 2ga(1 - \cos\theta)$$

or

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - mga(1 - \cos\theta) = \frac{1}{2}mu^2 - mgh \tag{3.2.13}$$

where $h$ is the vertical distance travelled by the particle. Equation (3.2.13) can be obtained directly from the principle of conservation of energy. Equation (3.2.12) then gives

$$T = m\frac{v^2}{a} + mg\cos\theta = m\frac{u^2}{a} - 2mg + 3mg\cos\theta$$

At the highest point $\theta = \pi$ and $T = m\frac{u^2}{a} - 5mg$.

If $u^2 \geqslant 5ag$, the particle will move in the complete vertical circle again and again.

However if $u^2 < 5ag$, tension will vanish before the particle reaches the highest point.

When the tension vanishes, the particle begins to move freely under gravity and describes a parabolic path till the string again becomes tight and the circular motion is started again. □

### 3.2.2 Motion of a Particle on a Smooth or Rough Vertical Wire

(a) If the particle moves on the inside of a smooth wire, the equations of motion (Fig. 3.5a) are:

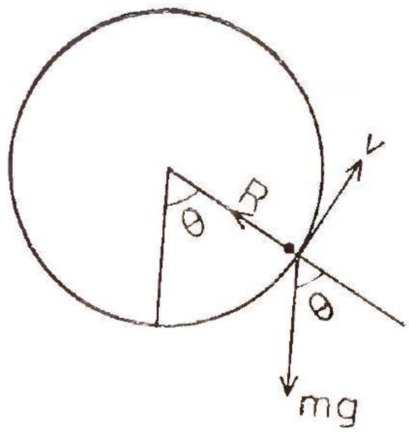
$$ma\theta'' = -mg\sin\theta \tag{3.2.14}$$

$$ma\theta'^2 = R - mg\cos\theta \tag{3.2.15}$$

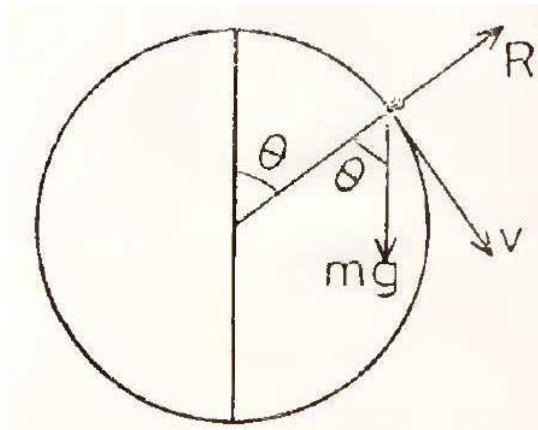
These are the same as (3.2.11) and (3.2.12) when $T$ is replaced by the normal reaction $R$.

As such if $u^2 \geqslant 5ag$, the particle makes an indefinite number of complete rounds of the circular wire.

If $u^2 < 5ag$, the reaction vanishes before the particle reaches the highest point, the particle leaves the curve, describes a parabolic path till it meets the circular wire again and it again describes a circular path.



(a)



(b)

Figure 3.5

This motion is repeated again and again.

(b) If the particle moves on the outside of the smooth vertical wire (Fig. 3.5b), the equations of motion are

$$ma\theta'' = mg\sin\theta \quad (3.2.16)$$

$$ma\theta'^2 = -R + mg\cos\theta \quad (3.2.17)$$

Integrating (3.2.16)

$$a\int\theta'' d\theta = g\int\sin\theta$$

$$a\theta' = -g\cos\theta + A$$

$\theta'^2 = u^2 + 2ga(1-\cos\theta)$

Using (3.2.17) $R = 3mg\cos\theta - mg - \frac{mu^2}{a}$

At the highets point $\theta = 0, R = mg - \frac{mu^2}{a}$

At the point $A$,

$$\theta = \pi/2, R = -\frac{mu^2}{a} - 2mg$$

If $u^2 > ag$, the particle leaves contact with the wire immediately and describes a parabolic path.

If $u^2 < ga$, the particle remains in contact for some distance, but leaves contact when $R$ vanishes i.e. before it reaches $A$ and then it describes a parabolic path.

(c) If the particle moves on the inside of rough vertical circular wire, then there is an additional frictional force $\mu R$ along the tangent opposing the motion.

As such equations (3.2.14) and (3.2.15) are modified to

$$\begin{aligned} ma\theta'' &= -mg\sin\theta - \mu R \\ ma\theta'^2 &= -mg\cos\theta + R \implies R = ma\theta'^2 + mg\cos\theta \end{aligned}$$

Eliminating $R$ between these equations, we get a non-linear differential equation

$$\begin{aligned} ma\theta'' &= -mg\sin\theta - \mu\left(mg\cos\theta + ma\theta'^2\right) \\ a\theta'' &= -g\sin\theta - \mu\left(g\cos\theta + a\theta'^2\right) \end{aligned}$$

which can be integrated by substituting $\theta' = w, \theta'' = wdw/d\theta$.

Similarly (3.2.16) and (3.2.17) are modified to

$$\begin{aligned} ma\theta'' &= mg\sin\theta - \mu R \\ ma\theta'^2 &= -R + mg\cos\theta \end{aligned}$$

We can again eliminate $R$, solve for $\theta'$ and $\theta$ and find the value of $\theta$ when $R$ vanishes.

□

### 3.2.3 Circular Motion of Satellites

Just as planets move in elliptic orbits with the Sun in one focus, the manmade artificial satellites move in elliptic (or circular) orbits with the Earth (or rather its centre) at one focus.

If the Earth is of mass $M$ and radius $a$ and a satellite of mass $m(\ll M)$ is projected from a point $P$ at a height $h$ above the Earth with velocity $V$ at right angles to $OP$ (Figure 3.6) it will move under a central force $GmM/r^2$.

Since the central force of a circular orbits is $mV^2/r$, we get, if the path is to be circular,

$$\frac{mV^2}{a+h} = \frac{GmM}{(a+h)^2} \quad \text{or} \quad V^2 = \frac{GM}{a+h} \tag{3.2.18}$$

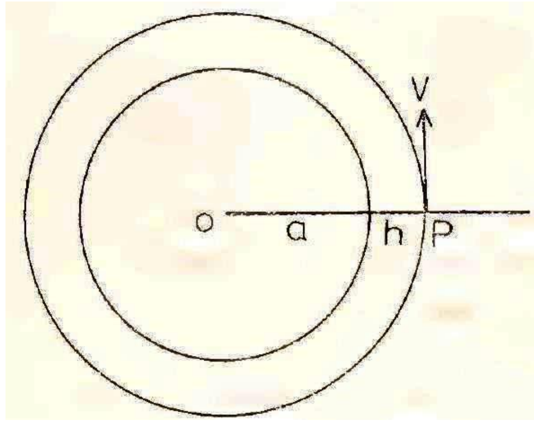



Figure 3.6

If $g$ is the acceleration due to gravity, then the gravitational force on a particle of mass $m$ on the surface of the Earth is $mg$.

Alternatively from Newton's inverse square law, it is $GMm/a^2$ so that

$$\frac{GMm}{a^2} = mg \quad \text{or} \quad GM = ga^2 \tag{3.2.19}$$

From (3.2.18) and (3.2.19)

$$V^2 = \frac{ga^2}{a+h}$$

This gives the velocity of a satellite describing a circular orbit at a height $h$ above the surface of the Earth. Its time period is given by

$$T = \frac{2\pi(a+h)}{V} = \frac{2\pi(a+h)}{\sqrt{ga}}(a+h)^{1/2} = \frac{2\pi}{\sqrt{ga}}(a+h)^{3/2}$$

The earth makes completes one revolution about its axis in twenty-four hours.

As such if $T$ is 24 hours, the satellite would have the same period as the Earth and would appear stationary, to an observer on the Earth.

Now taking $g = 32\text{ft/sec}^2, a = 4000$ miles, $T = 24$ hours, we get if $h$ is measured in miles

$$((4000+h)\times 1760\times 3)^{3/2} = \frac{24\times 60\times 60\sqrt{32}\times 4000\times 1760\times 3\times 7}{2\times 22}$$
$$= 1642607.416\times 10^6$$
$$(4000+h)\times 5280 = 13919.3408\times 10^4$$
$$4000+h = 26.36238788\times 10^3 = 26362.38788$$
$$h = 22362.38788\text{miles}$$

This gives the height of the synchronous or synchron satellite which is very useful for communication purposes. □

### 3.2.4 Elliptic Motion of Satellites

If a satellite is projected at a height $a+h$ above the centre of the Earth with a velocity different from $\sqrt{ga}/\sqrt{a+h}$ or if it is not projected at right angles to the radius vector, the orbit will not be circular, but can be elliptic, parabolic or hyperbolic depending on $V$ and the angle of projection.

If the angle of projection is 90° and the orbit is an elliptic with semi major axis $a'$ and eccentricity $e$, then there are two possibilities depending on whether the point of projection is the apogoes on the perigee

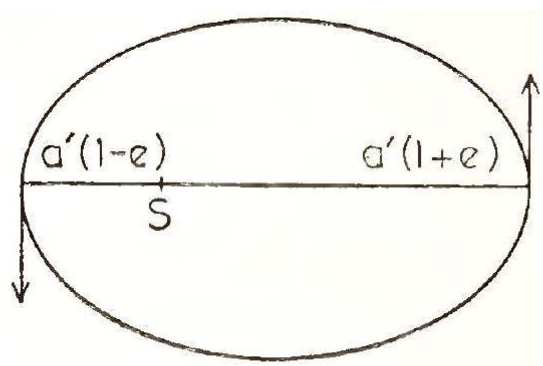



Figure 3.7

Using equation $V^2 = \mu(\frac{2}{r} - \frac{1}{a})$ in the case of ellipse,

$$V^2 = \mu\left(\frac{2}{a'(1+e)} - \frac{1}{a'}\right), a'(1+e) = a+h$$

or

$$V^2 = \mu\left(\frac{2}{a'(1-e)} - \frac{1}{a'}\right), a'(1-e) = a+h$$

i.e. $V^2 = \frac{ga^2}{a+h}(1-e)$ or $V^2 = \frac{ga^2}{a+h}(1+e)$

i.e. $V^2 = V_0^2(1-e)$ or $V^2 = V_0^2(1+e)$,

where $V_0$ is the velocity required for a circular orbit for which $e = 0$. Thus if $V > V_0$, the point of projection is nearest point of the orbit to the centre of the Earth and if $V < V_0$, this point is the furthest point.

For the elliptic orbit, the time period is

$$T = \frac{2\pi}{\sqrt{ga}} a'^{3/2}$$

where if $V < V_0, V^2 = V_0^2(1-e) \implies e = \sqrt{1 - \frac{V^2}{V_0^2}}$, and

$a'(1+e) = a+h \implies a' = \frac{a+h}{1+\sqrt{1-V^2/V_0^2}}$

and if $V > V_0$, $V^2 = V_0^2(1+e) \implies e = \sqrt{\frac{V^2}{V_0^2} - 1}$,, and

$a'(1-e) = a+h \implies a' = \frac{a+h}{1-\sqrt{V^2/V_0^2-1}}$ If $h_{\max}$ and $h_{\min}$ are the maximum and minimum heights of a satellite above the Earth's surface and $a$ is the radius of the Earth, we get

$$\frac{a'(1+e)}{a'(1-e)} = \frac{a+h_{\max}}{a+h_{\min}} \text{ or } \frac{1+e}{a+h_{\max}} = \frac{1-e}{a+h_{\min}}$$
$$= \frac{2}{2a+h_{\max}+h_{\min}}$$

or

$$\frac{1+e}{a+h_{\max}} = \frac{1}{a+\frac{h_{\max}+h_{\min}}{2}} = \frac{e}{\frac{h_{\max}-h_{\min}}{2}}$$
$$e = \frac{h_{\max}-h_{\min}}{2a+h_{\max}-h_{\min}}$$

# Let us sum up:

- Circular motion.
- Particle motion on a smooth or rough vertical wire.
- Circular and elliptic motion of satellite.

## Check your progress:

1. What is the time period of small oscillations of a simple pendulum?
2. Give the equations if the particle moves on the inside and outside of a smooth wire.
3. What is the time period for elliptic orbit ?

# 3.3 Linear Second Order Differential Equations

### 3.3.1 Rectilinear Motion

Let one end 0 of an elastic string of natural length $L(= 0A)$ be fixed (Figure 3.8) and let the other end to which a particle of mass $m$ is attached


Figure 3.8

be stretched a distance $a$ and then released. At any time $t$, let $x(t)$ be the extension, then the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = -\lambda\frac{x}{L} = -kx \tag{3.3.20}$$

where $k$ is the elastic constant. If the particle moves in a resisting medium with resistance proportional to the velocity $x'$,(3.3.20) becomes

$$mx'' + cx' + kx = 0 \tag{3.3.21}$$

which is a linear differential equation of the second order. Its solution is

$$x(t) = A_1e^{\lambda_1 t} + Ae^{\lambda_2 t}$$

where $\lambda_1, \lambda_2$ are the roots of

$$m\lambda^2 + c\lambda + k = 0$$

Here $\lambda_1 + \lambda_2 = -\frac{c}{m}, \lambda_1\lambda_2 = \frac{k}{m}$. The sum of the roots is negative and the product of the roots is positive.

Case (i) $c^2 > 4$ km, the roots are real and distinct and are negative. As such $x(t) \to 0$ as $t \to \infty$. The motion is said be overdamped.

Case (ii) $c^2 = 4$ km, the roots are real and equal and

$$x(t) = (A_1 + A_2 t)\exp\left(-\frac{c}{2m}t\right)$$

and again $x(t) \to 0$ as $t \to \infty$. In this case the motion is said to be critically damped.

Case (iii) $c^2 < 4$ km, the roots are complex conjugate with the real parts of the roots negative. $x(t)$ always oscillates but oscillations are damped out and tend to zero. In this case, the motion is said to be under damped.

Next we consider the case when there is an external force $m \cdot F(t)$ acting on the particle. In this case (3.3.21) becomes

$$mx'' + cx' + kx = mF(t) \tag{3.3.22}$$

A particular case of interest is given by the model

$$x'' + w_0^2 x = F\cos wt \tag{3.3.23}$$

i.e., when in the absence of the external force, the motion is simple harmonic with period $2\pi/w_0$ and the external force is periodic with period $2\pi/w$. The solution of (3.3.23) is given by

$$A.E\ is\ m^2 + w_0^2 = 0$$

$$m = \pm iw$$

$$C.F \ = \ A\cos w_0 t - B\sin w_0 t$$

$$P.I \ = \ \tfrac{F\cos wt}{D^2 - w_0^2}$$

$$= \tfrac{F\cos wt}{w^2 - w_0^2}\ if\ w \neq w_0$$

$$= \frac{tF\cos wt}{2D} \; if \; w = w_0$$

$$\Longrightarrow \qquad = \frac{F\cos wt}{w^2 - w_0^2} \; if \; w \neq w_0$$

$$= \frac{F}{2w_0} t \sin w_0 t \; if \; w = w_0$$

Hence

$$x(t) = A\cos(w_0 t - \alpha) + F\cos w_t t \left(w_0^2 - w^2\right) \quad w \neq w_0$$

$$= A\cos(w_0 t - \alpha) + \frac{F}{2w_0} t \sin w_0 t \qquad w = w_0$$

When $w = w_0$, the first term is periodic and its amplitude never exceeds $|A|$. However as $t \to \infty$ along a sequence for which $\sin \mu_0 t = \pm 1$, the magnitude of the second term approaches infinity.

The phenomenon we have discussed here is known as of pure or undamped resonance. It occurs when $c = 0$ and the input and natural frequencies are equal. We shall get a similar phenomenon when $c$ is small. The forcing function $F\cos wt$ is then said to be in resonance with the system.

Bridges, cars, planes, ships are vibrating systems and an external periodic force with the same frequency as their natural frequency can damage them. This is the reason why soldiers crossing a bridge are not allowed to march in step. However resonance phenomenon can also be used to advantage e.g. in uprooting trees or in getting a car out of a ditch.

When $w$ and $w_0$ differ only slightly, the solution represents superposition of two sinusoidal waves whose periods differ only slightly and this leads to the occurrence of beats. □

### 3.3.2 Electrical Circuits

Figure 3.9 shows an electrical circuit. The current $i(t)$ amperes represents the time rate of change of charge $q$ flowing in the circuits, so that

$$\frac{dq}{dt} = i(t) \tag{3.3.24}$$

(i) There is a resistance of $R$ Ohms in the circuit. This may be provided by a light bulb, an electric heater or any other electrical device opposing the motion of the charge and causing a potential drop of magnitude $E_R = Ri$ volts.

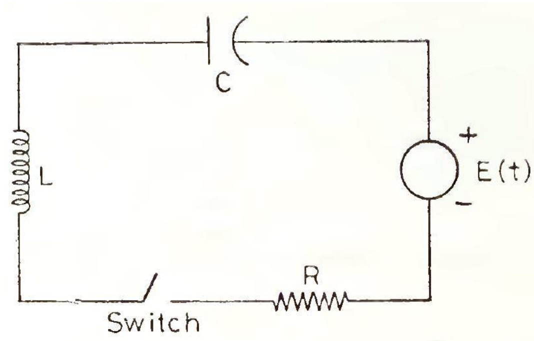



Figure 3.9

(ii) There is an induction of inductance $L$ henrys which produces a potential drop $E_L = L$ dildt.

(iii) There is a capacitance $C$ which produces a potential drop

$$E_c = \frac{1}{C}q$$

All these potential drops are balanced by the battery which produces a voltage $E$ volts. Now according to Kirchhoff's second law, the algebraic sum of the voltage drops round a closed circuit is zero so that

$$Ri + L\frac{di}{dt} + \frac{1}{C}q = E(t) \tag{3.3.25}$$

Differentiating and using (3.3.24), we get

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dE}{dt}. \tag{3.3.26}$$

Also substituting (3.3.24) in (3.3.25) we get

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t) \tag{3.3.27}$$

Both (3.3.26) and (3.3.27) represent linear differential equations with constant coefficients and their solutions will determine $i(t)$ and $q(t)$.

Comparing (3.3.22) and (3.3.27), we get the correspondences

mass m $\leftrightarrow$ inductance L

$$\text{friction coefficient } c \leftrightarrow \text{resistance } R$$

$$\text{spring constant } k \leftrightarrow \text{inverse capacitance } 1/C$$

$$\text{impressed force } F \leftrightarrow \text{impressed voltage } E$$

$$\text{displacement } x \leftrightarrow \text{charge } q$$

$$\text{velocity } v = dx/dt \leftrightarrow \text{current } i = \frac{dq}{dt}.$$

This shows the correspondence between mechanical and electrical systems. This forms the basis of analogue computers.

A linear differential equation of the second order can be solved by forming an electrical circuit and measuring the electric current in it. Similar analogues exist between hydrodynamical and electrical systems.

Mathematical modelling brings out the isomorphisms between mathematical structures of quite different systems and gives a method for solving all these models in terms of the simplest of these models.

We can have analogues of (3.3.22), (3.3.27) in economic system when $k(t)$ represents the excess of the capital invested over the equilibrium capital and $E(t)$ can represent external investments. □

### 3.3.3 Phillip's Stabilization Model for a Closed Economy

The assumptions of the model are:

(i) The producers adjust the national production $Y$ of a product according to the aggregate demand $D$.

If $D > Y$, they increase production and if $D < Y$, they decrease production so that we get

$$dY \mid dt = \alpha(D - Y), \alpha > 0 \tag{3.3.28}$$

where $\alpha$ is a reaction coefficient representing the velocity of adjustment.

(ii) Aggregate demand $D$ is the sum of private demand, government demand $G$ and an exogenous disturbance $u$.

The private demand in proportional to the national income or output so that

$$D = (1 - L)Y + G - u \tag{3.3.29}$$

where $1 - L$ is the marginal propensity to spend i.e. it is the marginal propensity to consume plus the marginal propensity to invest. We assume that $0 < L < 1$.

(iii) The government adjusts its demand to bring the national out-put to a desired level, which without loss of generality may be taken as zero.

The Government decides its demand according to one of the following policies:

(a) proportionate stabilization policy according to which

$$G^* = -f_p Y \tag{3.3.30}$$

where $f_p > 0$ is the coefficient of proportionality and we use the negative sign on the right hand side since if the output is less than the described level, government will come out with a positive demand.

(b) derivative stabilization policy according to which

$$G^* = -f_d Y', \tag{3.3.31}$$

where $f_d > 0$ and the government demand is proportional to $Y'$.

(c) mixed proportionate derivative policy according to which

$$G^* = -f_p Y - f_d Y' \tag{3.3.32}$$

(d) integral stabilization policy according to which

$$G^* = -f_I \int_0^t Y dt, \quad f_I > 0 \tag{3.3.33}$$

(iv) $G^*$ is the potential demand which the Government may like to make, but the actual demand $G$ will be gradually adjusted so that

$$G' = \beta (G^* - G) \tag{3.3.34}$$

where $\beta$ is the reaction coefficient. $\beta > 0$ since if $G < G^*$, the government tends to increase the demand to reach $G^*$.

Now from (3.3.28) and (3.3.29)

$$dY/dt = \alpha((1-L)Y + G - u - Y) \tag{3.3.35}$$

$$\implies \quad G = \frac{dY/dt}{\alpha} + (LY + u)$$

so that

$$d^2Y/dt^2 = \alpha dY/dt - \alpha LdY/dt + \alpha dG/dt - \alpha dY/dt$$

$$d^2Y/dt^2 = -\alpha LdY/dt + \alpha dG/dt \tag{3.3.36}$$

Eliminating $G$ between (3.3.34), (3.3.35) and (3.3.36)

$$\frac{d^2Y/dt^2}{\alpha} + LdY/dt = \beta\,(G^* - G)$$

$$\frac{d^2Y/dt^2}{\alpha} + LdY/dt = \beta\left(G^* - \frac{dY/dt}{\alpha} - (Ly + u)\right) \tag{3.3.37}$$

or

$$d^2Y/dt^2 + dY/dt(\alpha L + \beta) + \alpha BLY + \alpha\beta u = \alpha\beta G^* \tag{3.3.38}$$

If we substitute for $G^*$ from (3.3.30), (3.3.31) or (3.3.32), we get a linear differential equation of the second order with constant coefficients.

If however the government uses integral stabilization policy, we use (3.3.33) to get the third order differential equation

$$d^3Y/dt^3 + (\alpha 1 + \beta)d^2Y/dt^2 + \alpha\beta dY/dt + \alpha\beta f_I Y = 0 \tag{3.3.39}$$

The equations (3.3.38) and (3.3.39) can be easily solved. Even without solving these, the stability of the solutions and their behaviour as $t \to \infty$ can be easily obtained. □

# Let us sum up:

- Rectilinear motion.

- Electrical circuit.
- Phillips's Closed-economy stabilization model.

# Check your progress:

1. What is the solution of the differential equation of a the particle moves in a resisting medium with resistance proportional to the velocity $x'$?
2. What are the policies that deals Phillip's Stabilization Model for a Closed Economy?

## 3.4 Miscellaneous Mathematical Models

### 3.4.1 The Catenary

**Problem 3.4.1.** *Derive the equation of Catenary using mathematical modelling.*

**Solution**. A perfectly inflexible string is suspended under gravity from two fixed points $A$ and $B$ (Fig. 3.10).

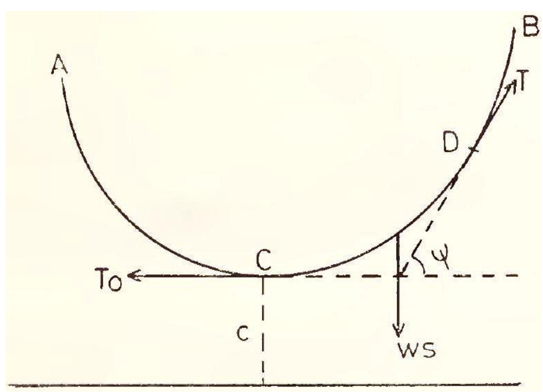



Figure 3.10

Consider the equilibrium of the part $CD$ of the string of length $s$ where $C$ is the lowest point of the string at which the tangent is horizontal.

The forces acting on this part of the string are (i) tension $T_0$ at $C$ (ii) tension $T$ at point $D$ along tangent at $D$ (iii) weight $ws$ of the string.

Equating the horizontal and vertical components of forces, we get

$$T\cos\psi = T_0, \quad T\sin\psi = ws \tag{3.4.40}$$

Let $T_0$ be equal to weight of length $c$ of the string, then (3.4.40) give

$$\tan\psi = \tfrac{ws}{To} = \tfrac{ws}{wc} = \tfrac{s}{c}$$
$$\tfrac{ds}{d\psi} = \rho = c\sec^2\psi,$$

where $\rho$ is radius of curvature of the string at $D$; so that

$$\begin{aligned}\frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}} &= c\left(1+\left(\frac{dy}{dx}\right)^2\right) \\ c\left(\frac{d^2y}{dx^2}\right) &= \left(1+\left(\frac{dy}{dx}\right)^2\right)^{1/2},\end{aligned} \tag{3.4.41}$$

which is a non-linear differential equation of second order. If $\frac{dy}{dx} = p$, then (3.4.41) gives

$$c\frac{dp}{\sqrt{1+p^2}} = dx$$

Integrating

$$\int \frac{dp}{\sqrt{1+p^2}} = \frac{1}{c}\int dx$$

$$\sinh^{-1} p = \frac{x}{c} + A$$

When $x = 0, p = 0$, so that $A = 0$ and

$$\frac{dy}{dx} = \sinh\frac{x}{c}$$

Integrating

$$y = c\cosh\frac{x}{c},$$

where we choose $x$-axis in such a way that $y = c$ when $x = 0$. This is the equation of the common catenary.

It may be noted that here we get a differential equation of the second order from a problem of statics rather than from a problem of dynamics. □

### 3.4.2 A Curve of Pursuit

**Problem 3.4.2.** *Explain a curve of pursuit with example through mathematical modelling.*

**Solution**. A ship at the point $(a, 0)$ sights a ship at $(0, 0)$ moving along $y$-axis with a uniform velocity $ku(0 < k < 1)$. It begins to pursue ship $B$ with a velocity $u$ always moving in the direction of the ship $B$ so that at any time $AB$ is along the tangent to the path of $A$.

From Figure 3.11

$$\begin{aligned} \tan(\pi - \psi) &= \frac{kut - y}{x} \\ -\frac{dy}{dx} &= -\frac{y}{x} + \frac{kut}{x} \\ x\frac{dy}{dx} - y &= -kut \end{aligned} \tag{3.4.42}$$

Differentiating with respect to $x$, we get

$$x\frac{d^2y}{dx} = -ku\frac{dt}{dx} \tag{3.4.43}$$

Now $dx/dt =$ horizontal component of velocity of $A = u\cos(\pi - \psi)$

$$= -u\cos\psi = -\frac{u}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

so that from (3.4.42) and (3.4.43)

$$x\frac{d^2y}{dx^2} = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Putting $\frac{dy}{dx} = p$, we get

Integrating

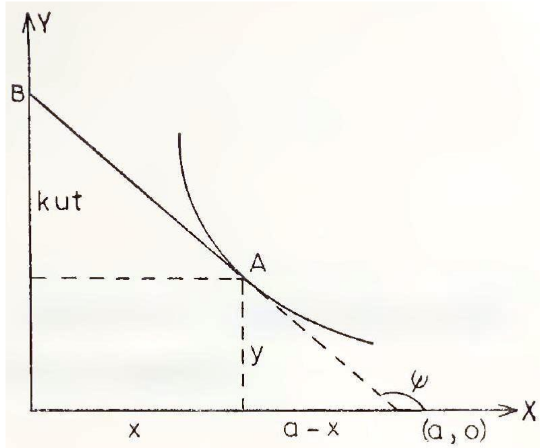



Figure 3.11

$$\frac{1}{k}\int \frac{dp}{\sqrt{1+p^2}} = \int \frac{dx}{x}$$
$$\frac{1}{k}\sinh(p) = \log x - \log a$$
$$\frac{dy}{dx} = k\left(\sinh^{-1}\left(\ln\frac{x}{a}\right)\right)$$

Integrating once again, we get $y$ as a function of $x$.

## Let us sum up:

- The catenary.
- A curve of pursuit.

## Check your progress:

1. Explain The Catenary and a curve of pursuit.

## Summary:

In this unit, we have modelled and analysed the planetary motions, circular motion and motion of satellites. In addition, to modelled planetary motions through linear differential equations of second order. Finally, we solved simple problems in second-order ordinary differential equations.

## Glossary:

Velocity and acceleration vectors, Radial and transverse direction, Central force, Inverse square law, Kepler's laws, Circular motion.

## Self Assessment Questions

1. What are the types of Mathematical modelling of Planetary motion?
2. Find the value of g at the surface of the Sun.
3. Show that the force required to make a particle of mass move in a circular orbit of radius a with velocity $\nu$ is $m\nu^2/a$ directed towards the centre.
4. Solve $x'' + 13x' + 36x = 0; x(0) = 1, x'(0) = 0$ and plot $x(t)$ against $t$ .

## Exercises

1. Find the central force $F(r)$ if the orbit is an ellipse with the centre of force coinciding with the centre of the ellipse.
2. Complete the discussion of motion of a particle on the inside of a smooth vertical circular wire when it is projected from the lowest period with horizontal velocity $2\sqrt{ag}$.
3. Solve $x'' + 8x' + 36x = 24\cos 6t$ and discuss the behaviour of the solution as t approaches infinity.
4. Obtain the curves of pursuit when $k = 1, k > 1$.

## Answers for check your progress

### Section 3.1

1. $r'' - r\theta'^2$ and $\frac{1}{r}\frac{d}{dt}\left(r^2\theta'\right)$

2. (i) Every planet describes an ellipse with the Sun at one focus (ii) The radius vector from the Sun to a planet describes equal areas in equal intervals of time.(iii) The squares of periodic time of planets are proportional to the cubes of the semi major axes of the orbits of the planets

### Section 3.2

1. $T = 2\pi\sqrt{a/g}$

2. If the particle moves on the inside of a smooth wire, the equations of motion are $ma\theta'' = -mg\sin\theta, ma\theta'^2 = R - mg\cos\theta$. If the particle moves on the outside of the smooth vertical wire, the equations of motion are $ma\theta'' = mg\sin\theta, ma\theta'^2 = -R + mg\cos\theta$.

3. $T = \frac{2\pi}{\sqrt{ga}}a'^{3/2}$

### Section 3.3

1. $x(t) = e^{\lambda_1 t} + e^{\lambda_2 t}$

2. Proportionate stabilization policy, Derivative stabilization policy, Mixed proportionate derivative policy, Integral stabilization policy.

### Section 3.4

1. Refer Section 3.4

## References:

1. J.N. Kapur, Mathematical Modelling, Wiley Eastern Limited, New Delhi, 4th Reprint, May 1994.

## Suggested Reading:

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2. A.C. Fowler, Mathematical Models in Applied Sciences, Cambridge University Press, 1997.

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4. Edward A. Bender, Introduction to Mathematical Modelling, Dover Publications, 1st ed., 2000.

# UNIT - 4

# Unit 4

# Mathematical Modelling Through Difference Equations

## Objectives:

- Introduce simple models through difference equations.
- Recall the basic theory of linear difference equations with constant coefficients.
- To develop models in economics and finance ? population dynamics and genetics.
- To solve simple problems.

## 4.1 The Mathematical Modelling through Mifference Equations

We need difference equation models when either the independent variable is discrete or it is mathematically convenient to treat it as a discrete variable. Thus in genetics, the genetic characteristics change from generation to generation and the variable representing a generation is a discrete variable.

In Economics, the price changes are considered from year to year or from month to month or from week to week or from day to day. In every case, the time variable is discretized.

In Population Dynamics, we consider the changes in population from one age-group to another and the variable representing the age-group is a discrete variable.

In finding the probability of $n$ persons in a queue or the probability of $n$ persons in a state or the probability of $n$ successes in a certain number of trials, the independent variable is discrete.

For mathematical modelling through differential equations, we give an increment $\Delta_x$ to independent variable $x$, find the change $\Delta y$ in $y$ and let $\Delta x \rightarrow 0$ to get differential equations. In most cases, we cannot justify the limiting process rigorously. Thus for modelling fluid motion, making $\Delta x \rightarrow 0$ has no meaning since a fluid consists of a large number of particles and the distance between two neighbouring particles cannot be made arbitrary small. Continuum mechanics is only an approximation (through fortunately a very good one) to reality.

Even if the limiting process can be justified e.g. when the independent variable is time, the resulting differential equation may not be solvable analytically. We then solve it numerically and for this purpose, we again replace the differential equation by a system of difference equations. Numerical methods of solving differential equations essentially mean solving difference equations.

It is even argued that since in most cases, we have to ultimately solve difference equations, we may avoid modelling through differential equations altogether. This is of course going too far since as we have seen in earlier chapters, mathematical modelling through differential equations is of immense importance to science and technology. Another argument in favour of difference equation models is that those biological and social scientists who do not know calculus and transcendental numbers like $e$ can still work with difference equation models and some important consequences of these models can be deduced with the help of even pocket calculators by even high school students.

We now give simple difference equation models parallel to the differential equation models studied in earlier chapters.

(i) Population Growth Model: If the population at time $t$ is $x(t)$, then assuming that the number of births and deaths in the next unit interval of time are proportional to the populations at time $t$, we get the model:

$$x(t+1) - x(t) = bx(t) - dx(t) \text{ or } x(t+1) = ax(t)$$

so that

$$x(t) = ax(t-1) = a^2x(t-2) = a^3x(t-3) = \ldots = a^t x(0)$$

This may be compared with the differential equation model:

$$\frac{dx}{di} = ax \text{ with the solution } x(t) = x(0)e^{at}$$

For solving the difference equation model, we require only simple algebra, but for solving the differential equation model, we require knowledge of calculus, differential equation and exponential functions.

(ii) Logistic Growth Model: This is given by

$$x(t+1) - x(t) = ax(t) - bx^2(t)$$

This is not easy to solve, but given $x(0)$, we can find $x(1), x(2), x(3), \ldots$ in succession and we can get a fairly good idea of the behaviour of the model with the help of a pocket calculator.

(iii) Prey-Predator Model: This is given by

$$\left.\begin{aligned} x(t+1) - x(t) &= -ax(t) + bx(t)y(t) \\ y(t+1) - y(t) &= py(t) - qx(t)y(t) \end{aligned}\right] \begin{aligned} a, b > 0 \\ p, q > 0 \end{aligned}$$

and again given $x(0), y(0)$, we can find $x(1), y(1); x(2), y(2); x(3), y(3), \ldots$, in succession.

(iv) Competition Model: This is given by

$$\left.\begin{aligned} x(t+1) - x(t) &= ax(t) - bx(t)y(t) \\ y(t+1) - y(t) &= px(t) - qx(t)y(t) \end{aligned}\right] \begin{aligned} a, b > 0 \\ p, q > 0 \end{aligned}$$

(v) Simple Epidemics Model: This is given by

$$\left.\begin{aligned} x(t+1) - x(t) &= -\beta x(t)y(t) \\ y(t+1) - y(t) &= \beta x(t)y(t) \end{aligned}\right], \quad \beta > 0$$

□

## Check your progress:

1. Give simple difference equation models parallel to the differential equation models studied.

## 4.2 Linear Difference Equations with Constant Coefficients

This theory is parallel to the corresponding theory of linear differential equations with constant coefficients, but is not usually taught in many places. We are therefore including a brief account here.

### 4.2.1 The Linear Difference Equation

An equation of the form

$$f\left(x_{t+n}, x_{t+n-1}, \ldots, x_t, t\right) = 0$$

is called a difference equations of $n$th order. The equation

$$f_0(t)x_{t+n} + f_1(t)x_{t+n-1} + \ldots + f_n(t)x_t = \varphi(t)$$

is called a linear difference equation, since it involves $x_t, x_{t+1}, \ldots, x_{t+n}$ only in the first degree.

The equation

$$a_0x_{t+n} + a_1x_{t+n-1} + \ldots + a_nx_t = \varphi(t) \tag{4.2.1}$$

is called a linear difference equation with constant coefficients.

The equation

$$a_0x_{t+n} + a_1x_{t+n-1} + \ldots + a_nx_t = 0 \tag{4.2.2}$$

is called a homogeneous linear difference equations with constant coefficients.

Let $x_t = g_1(t), g_2(t), \ldots, g_n(t)$ be $n$ linearly independent solutions of (4.2.2), then it is easily seen that

$$x_t = A_1g_1(t) + A_2g_2(t) + \ldots + A_ng_n(t)$$

is also a solution of (4.2.2) where $A_1, A_2, \ldots, A_n$ are $n$ arbitrary constants. This is the most general solution of (4.2.2).

Again it can be shown that if $G_1(t)$ is the solution of (4.2.2) containing $n$ arbitrary constants and $G_2(t)$ is any particular solution of (4.2.1) containing no arbitrary constant, then $G_1(t) + G_2(t)$ is the most general solution of (4.2.1), $G_1(t)$ is called the complementary function and $G_2$ is called a particular solution. □

### 4.2.2 The Complementary Function

**Problem 4.2.1.** *Describe the Complementary Function for the difference equation of nth order* $a_0x_{t+n} + a_1x_{t+n-1} + \ldots + a_nx_t = 0$.

**Solution**

We try the solution $x_t = a\lambda^t$. If this satisfies (4.2.2), we get

$$g(\lambda) \equiv a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n = 0$$

This algebraic equation of $n$th degree has $n$ roots $\lambda_1, \lambda_2, \ldots, \lambda_n$, real or complex. The complementary function is then given by

$$G_1(t) = c_1\lambda_1^t + c_2\lambda_2^t + \ldots + c_n\lambda_n^t \tag{4.2.3}$$

Case (i): If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all real and distinct, (4.2.3) gives us the complementary function when $c_1, c_2, \ldots, c_n$ are any $n$ arbitrary real constants.

Case (ii): If two of the roots $\lambda_1, \lambda_2$ are equal, then (4.2.3) contains only $n-1$ arbitrary constants and as such it cannot be the most general solution. We try the solution $ct\lambda_1^t$. We get

$$a_0(t+n)\lambda_1^n + a_1(t+n-1)\lambda_1^{n-1} + \ldots + a_n = 0$$
$$or \;\; \mathrm{t}g\,(\lambda_1) + g'\,(\lambda_1) = 0,$$

which is identically satisfied since both $g\,(\lambda_1) = 0$ and $g'\,(\lambda_1) = 0$ as $\lambda_1$ is a repeated root. In this case

$$G_1(t) = (c_1 + c_2 t)\,\lambda_1^t + c_3\lambda_3^t + c_4\lambda_4^t + \ldots + c_n\lambda_n^t$$

Case (iii): If a root $\lambda_1$ is repeated $k$ times, the complementary function is

$$G_1(t) = \left(c_1 + c_2 t + c_3 t^2 + \ldots + c_k t^{k-1}\right)\lambda_1^t + c_{k+1}\lambda_{k+1}^t + \ldots + c_n\lambda_n^t$$

Case (iv): Let $g(\lambda) = 0$ have two complex roots $\alpha \pm i\beta$, then their contribution to complementary function is

$$c_1(\alpha + i\beta)^t + c_2(\alpha - i\beta)^t$$

Putting $\alpha = r\cos\theta, \beta = r\sin\theta$ and using De Moivre's theorem, this reduces to

$$\begin{aligned} c_1 r^t(\cos\theta + i\sin\theta)^t + c_2 r^t(\cos\theta - i\sin\theta)^t &= r^t\cos(\theta t)\,(c_1 + c_2) + r^t\sin(\theta t)\,(ic_1 - ic_2) \\ &= r^t\,(d_1\cos(\theta t) + d_2\sin(\theta t)) \\ &= \left(\alpha^2 + \beta^2\right)^{t/2}(d_1\cos(\theta t) + d_2\sin(\theta t))\,, \end{aligned}$$

where $\tan\theta = \frac{\beta}{\alpha}$ and $d_1, d_2$ are arbitrary constants.

Case (v): If the complex roots $\alpha \pm i\beta$ are repeated $k$ times, then contribution to the complementary function is

$$\left(\alpha^2 + \beta^2\right)^{t/2}\left(\left(d_0 + d_1 t + \ldots + d_{k-1}t^{k-1}\cos(\theta t)\right) + \left(f_0 + f_1 t + \ldots + f_{k-1}t^{k-1}\right)\sin(\theta t)\right)$$

where $d_0, d_1, \ldots, d_{k-1}, f_0, \ldots, f_{k-1}$ are $2k$ arbitrary constants. □

### 4.2.3 The Particular Solution

Here we want a solution of (4.2.1) not containing any arbitrary constant.

Case (i): Let $\varphi(t) = AB^t, B$ is not a root of $g(\lambda) = 0$

We try the solution $CB^t$. Substituting in (4.2.1), we get

$$CB^t \left(a_0B^n + a_1B^{n-1} + \ldots + a_n\right) = AB^t$$

If $B \neq \lambda_1, \lambda_2, \ldots, \lambda_n$, we get

$$C = \frac{A}{a_0B^n + a_1B^{n-1} + \ldots + a_n}$$

and the particular solution is

$$\frac{AB^t}{a_0B^n + a_1B^{n-1} + \ldots + a_n}$$

Case (ii): Let $\varphi(t) = AB^t, B$ is a non-repeated root of $g(\lambda) = 0$

We try the solution $CtB^t$. Substituting in (4.2.1), we get

$$B^t \left(Ctg(B) + Cg'(B)\right) = AB^t$$

Since $g(B) = 0, g'(B) \neq 0$

$$C = \frac{A}{g'(B)}$$

so that the particular solution is

$$\frac{AtB^t}{a_0nB^{n-1} + a_1(n-1)B^{n-2} + \ldots + a_{n-1}}$$

Case (iii): Let $\varphi(t) = AB^t, \quad g(B) = 0, \quad g'(B) = 0, \ldots, g^{(k-1)}(B) = 0, \quad g^{(k)}(B) \neq 0$ then the particular solution is

$$\frac{At^{k-1}B^t}{g^{(k)}(B)}$$

Case (iv): Let

$$\varphi(t) = At^k$$

We try the solution

$$d_0 t^k + d_1 t^{k-1} + d_2 t^{k-2} + \ldots + d_k$$

Substituting in (4.2.1) we get

$$\begin{aligned} & a_0 \left(d_0(t+n)^k + d_1(t+n)^{k-1} + d_2(t+n)^{k-2} + \ldots + d_k\right) \\ & + a_1 \left(d_0(t+n-1) + d_1(t+n-1)^{k-1} + d_2(t+n-1)^{k-2} \right. \\ & \left. + \ldots + d_k\right) + \ldots + a_n \left(d_0 t^k + d_1 t^{k-1} + d_2 t^{k-2} + \ldots + d_k\right) \\ & = 0 \end{aligned}$$

Equating the coefficients of $t^k, t^{k-1}, \ldots, t^0$, on both sides, we get $(k+1)$ equations which in general will enable us to determine $d_0, d_1, d_2, \ldots, d_k$ and thus the particular solution will be determined. □

### 4.2.4 Obtaining Complementary Function by Use of Matrices

Let

$$\begin{aligned} x_t &= x_1(t) \\ x_{t+1} &= x_2(t) = x_1(t+1) \\ x_{t+2} &= x_3(t) = x_2(t+1) \\ &\ldots\ldots \\ x_{t+n} &= x_{n+1}(t) = x_n(t+1), \end{aligned} \tag{4.2.4}$$

so that $a_0 x_{t+n} + a_1 x_{t+n-1} + \ldots + a_n x_t = 0$ becomes

$$a_0 x_n(t+1) = -a_1 x_n(t) - a_2 x_{n-1}(t) - \ldots - a_n x_1(t) \tag{4.2.5}$$

Equations (4.2.4) and (4.2.5) give

$$x_1(t+1) = x_2(t)$$

$$
\begin{aligned}
x_2(t+1) =& x_3(t)\\
&\cdots\cdots\\
x_{n-1}(t+1) =& x_n(t)\\
x_n(t+1) =& -\frac{a_1}{a_0}x_n(t) - \frac{a_2}{a_0}x_{n-1}(t) - \cdots - \frac{a_n}{a_0}x_1(t),
\end{aligned}
$$

which can be written in the matrix form

$$
\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_n(t+1) \end{bmatrix}
$$

$$
\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & -\frac{a_{n-2}}{a_0} & \cdots & -\frac{a_1}{a_0} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ x_n(t) \end{bmatrix} \tag{38}
$$

or

$$
X(t+1) = AX(t) \tag{4.2.6}
$$

where $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$,

$$
A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ . & \cdots & \cdots & \cdots & \cdots \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & -\frac{a_{n-2}}{a_0} & \cdots & -\frac{a_1}{a_0} \end{bmatrix}
$$

Applying (4.2.6) repeatedly

$$
X(k) = A^k X(0)
$$

where

$$X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ . \\ x_n(0) \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ \vdots \\ x_1(n-1) \end{bmatrix} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

Thus knowing the values of $x_1$ at times $0, 1, 2 \ldots, n-1$, we can find its value at all subsequent times. □

### 4.2.5 Solution of a System of Linear Homogeneous Difference Equations with Constant Coefficients

Let the system be given by

$$x_1(t+1) = a_{11}x_1(t) + a_{12}x_2(t) + \ldots + a_{1n}x_n(t)$$

$$x_2(t+1) = a_{21}x_1(t) + a_{22}x_2(t) + \ldots + a_{2n}x_n(t)$$

$$x_n(t+1) = a_{n1}x_1(t) + a_{n2}x_2(t) + \ldots + a_{nn}x_n(t)$$

This can be written in the matrix form

where

$$X(t+1) = AX(t) \tag{4.2.7}$$

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ . \\ x_n(t) \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & . & a_{nn} \end{bmatrix}$$

Applying (4.2.7) repeatedly, we get

$$X(k) = A^k X(0)$$

□

### 4.2.6 Solution of Linear Difference Equations by Using Laplace Transform

Let the linear difference equation be

$$a_0 f(t) + a_1 f(t-1) + \ldots + a_n f(t-n) = \varphi(t)$$
$$f(t) = 0 \quad \text{when } t < 0$$

Let $\vec{f}(\lambda)$ be the Laplace transform of $f(t)$ so that

$$\bar{f}(\lambda) = L(f(t)) = \int_0^\infty e^{-\lambda t} f(t) dt$$

then $L(f(t-1)) = \int_1^\infty e^{-\lambda t} f(t-1) dt$

$$\begin{aligned} &= e^{-\lambda} \int_0^\infty e^{-\lambda t} f(t) dt = e^{-\lambda} \bar{f}(\lambda) \\ L(f(t-2)) &= \int_2^\infty e^{-\lambda t} f(t-2) dt \\ &= e^{-2\lambda} \int_0^\infty e^{-\lambda t} f(t) dt = e^{-2\lambda} \bar{f}(\lambda) \end{aligned} \tag{4.2.8}$$

and so on so that taking Laplace transform of both sides of (4.2.8), we get

$$\left(a_0 + a_1 e^{-\lambda} + a_2 e^{-2\lambda} + \ldots + a_n e^{-n\lambda}\right) \bar{f}(\lambda) = L(\varphi(t)) = \bar{\varphi}(\lambda),$$

so that $\bar{f}(\lambda)$ is known. Inverting the Laplace transform, we get $f(t)$. In this case $t$ is regarded as a continuous variate such that $f(t) = 0$ when $t < 0$. □

### 4.2.7 Solution of Linear Difference Equations by Using $z$-Transform

Let $\{u_n\}$ be an infinite sequence and $t$ be a discrete variate, then it is better to use the $z$-transform.

Hence its $z$-transform is defined by

$$Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$$

whenever this infinite series converges. If $\{u_n\}$ is a probability distribution and $z = 1/s$, it will be the same as the probability generating function.

The following results can be easily established

(i) If $k > 0, Z(u_{n-k}) = z^{-k} Z(u_n)$

(ii) If $k > 0, \quad Z(u_{n+k}) = z^k \left[Z(u_n) - \sum_{m=0}^{k-1} u_n z^{-m}\right]$

(iii) $u_n: \quad 1 \quad a^n \quad e^{an}$

$$Z(u_n): \quad z/(z-1) \quad z/(z-a) \quad z/\left(z-e^{\alpha}\right)$$

Taking $z$-transform of both sides of a linear difference equation, we can find $Z(u_n)$ and expanding it in powers of $1/z$ and finding the coefficient of $z^{-n}$, we can get $u_n$. □

### 4.2.8 Solution of non-Linear Difference Equations Reducible to Linear Equations

The equations

$$y_{n+1} = \sqrt{y_n}$$

$$y_n y_{n+2} = y_{n+1}^2$$

become linear on substitution $u_n = \ln y_n$ : Also

$$y_{n+2} = \frac{y_n y_{n+1}}{y_n + y_{n+1}}$$

becomes linear on substitution $u_n = 1/y_n$. □

### 4.2.9 Stability Theory for Difference Equations

If $x_t = K$ satisfies

$$f(x_t, x_{t+1}, x_{t+2}, \ldots, x_{t+n}) = 0 \tag{4.2.9}$$

then this gives an equilibrium position. To find its stability, we substitute $x_t = K + u_t$ in (4.2.9) and simplify neglecting squares and products and higher powers of $u_t$ 's to get a linear equation

$$a_1 u_{t+n} + a_2 u_{t+n-1} + \ldots + a_n u_t = 0$$

We try the solution $u_t = A\lambda^t$ and get the characteristic equation

$$a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0 \tag{4.2.10}$$

If the absolute value of each of the $n$ roots of this equation is less than unity, then $u_t$ would tend to zero as $t \to \infty$ for all small initial disturbances and the equilibrium position would be locally asymptotically stable.

The conditions for all the roots of (4.2.10) having magnitude less than unity are given by Schur's criterion viz. that all the following determinants should be positive.

$$\Delta_1 = \begin{vmatrix} a_0 & a_n \\ a_n & a_0 \end{vmatrix}, \qquad \Delta_2 = \begin{vmatrix} a_0 & 0 & \vdots & a_n & a_{n-1} \\ a_1 & a_0 & \vdots & 0 & a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & 0 & \vdots & a_0 & a_1 \\ a_{n-1} & a_n & \vdots & 0 & a_0 \end{vmatrix}$$

$$\Delta_n = \begin{vmatrix} a_0 & 0 & \cdots & 0 & \vdots & a_n & a_{n-1} & \cdots & a_1 \\ a_1 & a_0 & \cdots & 0 & \vdots & 0 & a_n & \cdots & a_2 \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot & \vdots & \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ a_{n-1} & a_n & \cdots & a_0 & \vdots & 0 & 0 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & 0 & \cdots & 0 & \vdots & a_0 & a_1 & \cdots & a_{n-1} \\ a_n & a_n & \cdots & 0 & \vdots & 0 & a_0 & \cdots & a_{n-2} \\ \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot & \vdots & \cdot\cdot & \cdot\cdot & \cdots & \cdot\cdot \\ a_1 & a_2 & \cdots & a_n & \vdots & 0 & 0 & \cdots & a_0 \end{vmatrix}$$

□

# Let us sum up:

- The linear difference equation.
- Using matrices to find complementary functions.

- Solving a system of linear difference equations with constant co efficient.
- Applying the Laplace transform to solve linear difference equations.
- Z-Transform for solving linear difference equations.
- Addressing non-linear difference equations convertible to linear formulas.
- Stability theory for difference equations.

## Check your progress:

1. What are the basic theory of Linear difference equations with constant coefficients?

## 4.3 Difference Equations in Economics and Finance

### 4.3.1 The Harrod Model

Let $S(t), Y(t), I(t)$ denote the savings, national income and investment respectively. We make now the following assumptions:

(i) Savings made by the people in a country depend on the national income i.e.

$$S(t) = \alpha Y(t), \quad \alpha > 0 \tag{4.3.11}$$

(ii) The investment depends on the difference between the income of the current year and the last year i.e.

$$I(t) = \beta(Y(t) - Y(t-1)), \quad \beta > 0 \tag{4.3.12}$$

(iii) All the savings made are invested, so that

$$S(t) = I(t) \tag{4.3.13}$$

From (4.3.11), (4.3.12) and (4.3.13), we get the difference equation

$$Y(t) = \frac{\beta}{\beta - \alpha} Y(t-1),$$

which has the solution

$$Y(t) = A\left(\frac{\beta}{\beta-\alpha}\right)^t = Y(0)\left(\frac{\beta}{\beta-\alpha}\right)^t$$

Assuming that $Y(t)$ is always positive,

$$\beta > \alpha, \beta/(\beta-\alpha) > 1,$$

so that the national income increases with $t$. The national incomes at different times $0, 1, 2, 3, \ldots$ form a geometrical progression.

Thus if all savings are invested, savings are proportional to national income and the investment is proportional to the excess of the current years income over the preceding years income, then the national income increases geometrically. □

### 4.3.2 The Cobweb Model

Let $p_t$ = price of a commodity in the year $t$ and $q_t$ = amount of the commodity available in the market in year $t$, then we make the following assumptions

(i) Amount of the commodity produced this year and available for sale is a linear function of the price of the commodity in the last year, i.e.

$$q_t = \alpha + \beta p_{t-1} \tag{4.3.14}$$

where $\beta > 0$ since if the last year's price was high, the amount available this year would also be high.

(ii) The price of the commodity this year is a linear function of the amount available this year i.e.

$$p_t = \gamma + \delta q_t \tag{4.3.15}$$

where $\delta < 0$, since if $q_t$ is large, the price would be low. From (4.3.14) and (4.3.15)

$$p_t - \beta\delta p_{t-1} = \gamma + \alpha\delta \tag{4.3.16}$$

which has the solution

$$\left(p_t - \frac{\alpha\delta + \gamma}{1 - \beta\delta}\right) = \left(p_0 - \frac{\alpha\delta + \gamma}{1 - \beta\delta}\right)(\beta\delta)^t$$

so that

$$\left(p_t - \frac{\alpha\delta + \gamma}{1 - \beta\delta}\right) = \left(p_{t-1} - \frac{\alpha\delta + \gamma}{1 - \beta\delta}\right)(\beta\delta)$$

Since $\beta\delta$ is negative $p_0, p_1, p_2, p_3, \ldots$ are alternatively greater and less than $(\alpha\delta + \gamma)/(1 - \beta\delta)$.

If $|\beta\delta| > 1$, the deviation of $p_t$ from $(\alpha\delta+\gamma)/(1-\beta\delta)$ goes on increasing. On the other hand if $|\beta\delta| < 1$, this deviation goes on decreasing and ultimately $p_t \to (\alpha\delta+\gamma)/(1-\beta\delta)$ as $t \to \infty$.

Figure 4.1 shows how the price approaches the equilibrium price $p_e = (\alpha\delta+\gamma)/(1-\beta\delta)$ as $t$ increases in the two cases when $p_0 > p_e$ and $p_0 < p_e$ respectively.

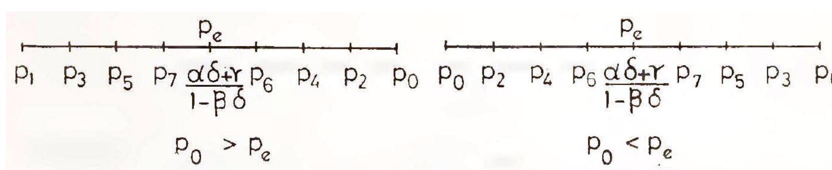


Figure 4.1

In the same way, eliminating $p_t$ from (4.3.14), (4.3.15) we get

$$q_t = \alpha + \beta\gamma + \beta\delta q_{t-1}$$

which has the solution

$$\left(q_t - \frac{\alpha + \beta\gamma}{1 - \beta\delta}\right) = \left(q_c - \frac{\alpha + \beta\gamma}{1 - \beta\delta}\right)(\beta\delta)^t$$

so that $q_t$ also oscillates about the equilibrium quantity level

$$q_t = (a + \beta\gamma)/(1 - \beta\delta) \text{ if } |\beta\delta| < 1$$

The variation of both prices and quantities is shown simultaneously in Figure 4.2.

Suppose we start in the year zero with price $p_0$, and quantity $q_0$ represented by the point $A$. In year 1 , the quantity $q_1$ is given by $\alpha + \beta p_0$ and the price is given by

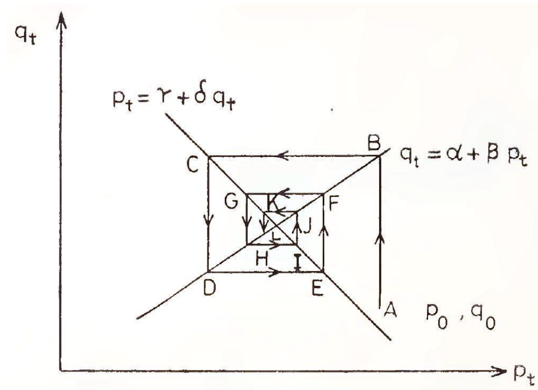


Figure 4.2

$p_1 = \gamma + \delta q_1$.

This brings us to the point $C$ in two steps via $B$. The path of prices and quantities is thus given by the Cobweb path $ABCDEFGHI, \ldots$ and the equilibrium price and quantity are given by the intersection of the two straight lines. □

### 4.3.3 Samuelson's Interaction Models

The basic equations for the first interaction model are:

$$Y(t) = C(t) + I(t), C(t) = \alpha Y(t-1), I(t) = \beta[C(t) - C(t-1)] \tag{4.3.17}$$

Here the positive constant $\alpha$ is the marginal propensity to consume with respect to income of the previous year and the positive constant $\beta$ is the relation given by the acceleration principle i.e. $\beta$ is the increase in investment per unit of excess of this years consumption over the last year's.

From (4.3.17), we get the second order difference equation

$$Y(t) - \alpha(1+\beta)Y(t-1) + \alpha\beta Y(t-2) = 0 \tag{4.3.18}$$

In the second interaction model, there is an additional investment by the government and this investment is assumed to be a constant $\gamma$. In this case (4.3.18) is modified to

$$Y(t) - \alpha(1+\beta)Y(t-1) + \alpha\beta Y(t-2) - \gamma = 0 \tag{4.3.19}$$

The solution of (4.3.18) and (4.3.19) can show either an increasing trend in $Y(t)$ or a decreasing trend in $Y(t)$ or an oscillating trend in it. $\square$

### 4.3.4 Application to Actuarial Science

One important aspect of actuarial science is what is called mathematics of finance or mathematics of investment.

If a sum $S_0$ is invested at compound interest of $i$ per unit amount per unit time and $S_t$ is the amount at the end of time $t$, then we get the difference equation

$$S_{t+1} = S_t + iS_t = (1+i)S_t$$

which has the solution

$$S_t = S_0(1+i)^t$$

which is the well-known formula for compound interest.

Suppose a person borrows a sum $S_0$ at compound interest $i$ and wants to amortize his debt, i.e. he wants to pay the amount and interest back by payment of $n$ equal instalments, say $R$, the first payment to be made at the end of the first year.

Let $S_t$ be the amount due at the end of $t$ years, then we have the difference equation

$$S_{t+1} = S_t + iS_t - R = (1+i)S_t - R$$

Its solution is

$$\begin{aligned} S_t &= \left(S_0 - \frac{R}{i}\right)(1+i)^t + \frac{R}{i} \\ &= S_0(1+t)^t - R\frac{(1+i)^t - 1}{i} \end{aligned}$$

If the amount is paid back in $n$ years, $S_n = 0$, so that

$$R = S_0 \frac{1}{1-(1+i)^{-n}} = S_0 \frac{1}{a_{\bar{n}}^{-i}}, \tag{4.3.20}$$

where $a_{\bar{n}|_i}$ called the amortization factor is the present value of an annuity of 1 per unit time for $n$ periods at an interest rate $i$.

The functions $a_{\bar{n}ii}$ and $(a_n)^{-1}$ are tabulated for common values of $n$ and $i$.

Suppose an amount $R$ is deposited at the end of every period in a bank and let $S_t$ be the amount at the end of $t$ periods, then

$$S_{t+1} = S_t(1+i) + R,$$

so that (since $S_0 = 0$ )

$$S_t = R\frac{(1+i)^t - 1}{i} = RS_{\overline{n|i}} \tag{4.3.21}$$

From (4.3.20) and (4.3.21)

$$\begin{aligned} S_{\overline{n|i}} &= (1+i)^n a_{\bar{n}|i} \\ or\ \frac{1}{S_{\overline{n|i}}} &= \frac{1}{a_{n|i}} + 1 \end{aligned} \tag{4.3.22}$$

If a person has to pay an amount $S$ at the end of $n$ years, he can do it by paying into a sinking find an amount $R$ per period where

$$R = S\frac{1}{S_{\overline{n|i}}}$$

where $\frac{1}{S_{\bar{n}i}i}$ is the sinking fund factor and can be tabulated by using (4.3.22). □

## Let us sum up:

- The Harrod model.
- The Cobweb model.
- Samuelson's interaction models.
- Utilizing actuarial science.

## Check your progress:

1. Explain the models through difference equations in finance and economics.

2. What are the assumptions made in the Harrod model?

3. What are the assumptions made in the cobweb model ?

# 4.4 Difference Equations in Population Dynamics and Genetics

### 4.4.1 Non-Linear Difference Equations Model for Population Growth Non-Linear Difference Equations

Let $x_t$ be the population at time $t$ and let births and deaths in time-interval $(t, t+1)$ be proportional to $x_t$, then the population $x_{t+1}$ at time $t+1$ is given by

$$x_{i+1} = x_t + bx_t - dx_t = x_t(1+a)$$

This has the solution

$$x_t = x_0(1+a)^t \tag{4.4.23}$$

so that the population increases or decreases exponentially according as $a > 0$ or $a < 0$.

We now consider the generalisation when births and deaths $b$ and $d$ per unit population depend linearly on $x_t$ so that

$$\begin{aligned} x_{t+1} &= x_t + (b_0 - b_1 x_t)\, x_t - (d_0 + d_1 x_t)\, x_t \\ &= mx_t - rx_t^2 = mx_t\left(i - \frac{r}{m}x_t\right) \end{aligned} \tag{4.4.24}$$

This is the simplest non-linear generalisation of (4.4.23) and gives the discrete version of the logistic law of population growth. However this model shows many new features not present in the continuous version of the logistic model. Let $r/mx_t = y_t$, then (4.4.24) becomes

$$y_{t+1} = my_t\,(1 - y_t) \tag{4.4.25}$$

□

**Problem 4.4.1.** *Explain the one-period, $2^n$-period, and other periods fixed points and their stability.*

**Solution**

### One-Period Fixed Points and Their Stability

A one-period fixed point of this equation is that value of $y_t$ for which $y_{t+1} = y_t$ i.e. for which

$$y_t = my_t\,(1 - y_t)\,,$$

so that there are two one-period fixed points 0 and $(m-1)/m$. If $y_0 = 0$, then $y_1, y_2, y_3, \ldots$ are all zero and the population remains fixed at zero value:

If $y_0 = (m-1)/m$, then $y_1, y_2, y_3, \ldots$ are all equal to $(m-1)/m$. The second fixed point exists only if $m > 1$.

We now discuss the stability of equilibrium of each of these equilibrium positions.

Putting $y_t = 0 + u_t$ in (4.4.25) and neglecting squares and higher powers of $u_t$, we get $u_{t+1} = mu_t$ and since $m > 0$, the first equilibrium position is one of unstable equilibrium.

Again putting $y_t = (m-1)/m + u_t$ in (4.4.25) and neglecting squares and higher powers of $u_t$, we get

$$u_{t+1} = (2 - m)u_t$$

so that the second position of equilibrium is stable only if $-1 < 2 - m < 1$ or if $1 > m - 2 > -1$ or if $1 < m < 3$.

Thus if $0 < m < 1$, there is only one one-period fixed point and it is unstable. If $1 < m < 3$, there are two one-period fixed points, the first is unstable and the second is stable. If $m > 3$, there are two one-period fixed points, both of which are unstable.

Two-Period Fixed Points and Their Stability

A point is called a two-period fixed point if it repeats itself after two periods i.e. if $y_{t+2} = y_t$ i.e. if

$$y_{t+2} = my_{t+i}\left(1 - y_{t+1}\right) = m^2 v_t \left(1 - y_t\right)\left(1 - my_t + my_i^2\right) = y_t$$

or

$$y_t\left(my_t - (m-1)\right)\left(m^2 y_t^2 - m(1+m)y_t + (1-m)\right) = 0$$

This is a fourth degree equation and as such there can be four two-period fixed points. Two of these are the same as the one-period fixed points. This is obvious from the consideration that every one-period fixed point is also a two-period fixed point. The genuine two-period fixed points are obtained by solving the equation

$$m^2 y_t^2 - m(1+m)y_t + (1+m) = 0$$

Its roots are real if $m > 3$. Thus if $m > 3$, the two one-period fixed points become unstable, but two new two-period fixed points exist and we can discuss their stability as before.

It can be shown that if $m_2 < m < m_4$, where $m_2 = 3$ and $m_4$ is a number slightly greater than 3 , then the two two-period fixed points are stable but if $m > m_4$, all the four one- and two periods become unstable, but four new four-period fixed points exist which are stable if $m_4 < m < m_8$ and become unstable if $m > m_8$.

### $2^n$-Period Fixed Points and Their Stability

It can be shown that there exists an increasing infinite sequence of real numbers $m_2$, $m_4$, $m_8, \ldots, m_2 n, m_2 n+1, \ldots$ such that when $m_2 n < m < m_2 n + 1$ there are $2^{n+1} 2^{n+1}$-period fixed points, out of which $2^n$ fixed points are also fixed points of lower order time periods and all these are unstable and the remaining $2^n$ points are genuine $2^{n+1}$ period fixed points and are stable.

From 4.3 represents the stable fixed period points.

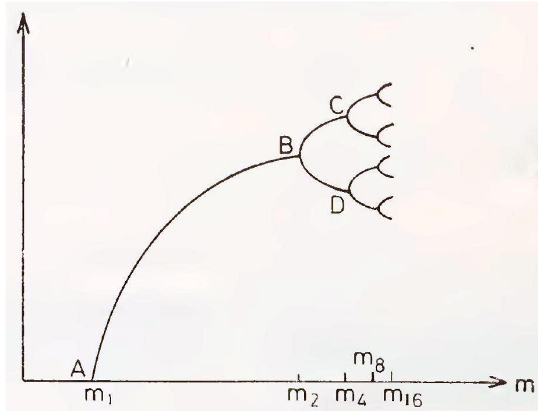


Figure 4.3

When $m$ lies between $m_1$ and $m_2$, there is one stable one-period fixed point.

When $m$ lies between $m_2$ and $m_4$ there are two stable two-period fixed points.

When $m$ lies between $m_4$ and $m_8$, there are four stable four-period fixed points, and so on.

### Fixed Points of other Periods

The sequence $m_2, m_4, m_8, \ldots$ is bounded above by a fixed number $m^*$. If $m > m^*$, there can be a three-period fixed point and if there is a threeperiod fixed point, there will also be fixed points of periods,

$$3, 5, 7, 9, \ldots$$

$$2 \cdot 3, 2 \cdot 5 \cdot 2 \cdot 7, 2 \cdot 9, \ldots$$

$$2^2 \cdot 3, 2^2 \cdot 5, 2^2 \cdot 7, \ldots$$

This is expressed by saying that Period Three Means Chaos.

### Chaotic Behaviour of the Non-linear Model

If $m$ lies between $m_8$ and $m_{16}$, there will be eight 16 -period stable fixed points. If a population size starts from any one of these values, it will oscillate through fifteen other values to return to the original value and this pattern will go on repeating itself. If we draw the graph, it will show rapid oscillations and will look like the graph representing a random phenomenon. Our model is perfectly deterministic, though its behaviour may appear to be random and stochastic. □

**Problem 4.4.2.** *What are the special features of non-linear difference equaton models*

**Solution**

**Special Features of Non-linear Difference Equation Models**

The simple model illustrates the differences in behaviour between difference and differential equation models. The problems of existence and uniqueness of solutions, of the stability of equilibrium positions are all different due to the basic fact that inspite of similarities, the Discrete and the Continuous are really different. □

### 4.4.2 Age-Structured Population Models

Let $x_1(t), x_2(t), \ldots, x_p(t)$ be the population sizes of $p$ pre-reproductive age-groups at time $t$

Let $x_{p+1}(t), x_{p+2}(t), \ldots, x_{p+q}(t)$ be the population sizes of $q$ reproductive age-groups at time $t$, and

Let $x_{p+q+1}(t), x_{p+q+2}(t), \ldots, x_{p+q+r}(t)$ be the population sizes of $r$ postreproductive age-groups at time $t$.

Let $b_{p+1}, b_{p+2}, \ldots, b_{p+q}$ be the birth rates i.e. the number of births per unit time per individual in the reproductive age groups.

In other age-groups, the birth rates are zero.

Let $d_1, d_2, \ldots, d_{p+q+r}$ be the death rates in the $p+q+r$ age-groups.

Let $m_1, m_2, \ldots, m_{p+q+r}$, be the rates of migration to the next age-groups, then we get the system of difference equations

$$
\begin{aligned}
&x_1(t+1) = b_{p+1}x_{p+1}(t) + \ldots + b_{p+q}x_{p+q}(t) - (d_1 + m_1)\, x_1(t) \\
&x_2(t+1) = m_1 x_1(t) - (d_2 + m_2)\, x_2(t) \\
&\cdots \quad \cdots \quad \cdots \\
&x_{p+q+r-1}(t+1) = m_{p+q+r-2}(t) - (d_{p+q+r-1} + m_{p+q+r-1})\, x_{p+q+r-1}(t) \\
&x_{p+q+r}(t+1) = m_{p+q+r-1}x_{p+q+r-1}(t) - (d_{p+q+r})\, x_{p+q+r}(t)
\end{aligned}
$$

which can be written in the matrix form

$$X(t+1) = LX(t) \tag{4.4.26}$$

where

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ x_{p+q+r}(t) \end{bmatrix}$$

$$L = \begin{bmatrix} -(d_1+m_1) & 0 & 0\ldots0 & b_{p+1} & b_{p+2}\ \ldots & b_{p+q} & 0 & .. & 0 & 0 \\ m_1 & -(d_2+m_2) & 0\ldots & 0 & 0\ \ldots & 0 & 0 & .. & 0 & 0 \\ 0 & m_2 & -(d_3+m_3) & 0 & 0\ \ldots & 0 & 0 & .. & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdot & & \cdot & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdot & & \cdot & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdot & & \cdot & \cdot \\ 0 & 0 & 0\ldots & 0 & 0 & 0 & 0 & .. & m_{n-1} & -d_n \end{bmatrix}$$

Figure 4.4

where $p+q+r=n$.

$L$ is called the Leslie matrix (refer Figure 4.4). All the elements of its main diagonal are negative and all the elements of its main subdiagonal are positive. In addition $q$ elements in the first row are positive and the rest of the elements are all zero. The solution of (4.4.26) can be written as

$$X(t) = L^t X(0)$$

Now the Leslie matrix has the property that it has a dominant eigenvalue which is real and positive, which is greater in absolute value than any other eigenvalue and for which the corresponding eigenvector has all its components positive.

If this dominant eigenvalue is greater than unity, then the population of all age-groups will increase exponentially and if it is less than unity the population of all age-groups will die out. If this dominant eigenvalue is unity, the population can have a stable age structure.

The Leslie model is in terms of a system of linear difference equations. If we take the effects of overcrowding and density dependence into account, the equations are nonlinear.

□

### 4.4.3 Mathematical Modelling through Difference Equations in Genetics

#### (a) Hardy-Weinberg Law

Every characteristic of an individual, like height or colour of the hair, is determined by a pair of genes, one obtained from the father and the other obtained from the mother.

Every gene occurs in two forms, a dominant (denoted by a capital letter say $G$ ) and a recessive (denoted by the corresponding small letter say $g$ ).

Thus with respect to a characteristic, an individual may be a dominant ($GG$), a hybrid ( $Gg$ or $gG$ ) or a recessive ($gg$ ).

In the $n$th generation, let the proportions of dominants, hybrids and recessives be $p_n, q_n, r_n$ so that

$$p_n + q_n + r_n = 1, \quad p_n \geqslant 0, q_n \geqslant 0, r_n \geqslant 0$$

We assume that individuals, in this generation mate at random.

Now $p_{n+1} =$ the probability that an individual in the $(n+1)$ th generation is a dominant $(GG) =$ (probability that this individual gets a $G$ from the father) $\times$ (probability that the individual gets a $G$ from the mother)

$$= \left(p_n + \frac{1}{2}q_n\right)\left(p_n + \frac{1}{2}q_n\right) = \left(p_n + \frac{1}{2}q_n\right)^2$$

or

$$p_{n+1} = \left(p_n + \frac{1}{2}q_n\right)^2 \tag{4.4.27}$$

Similarly

$$q_{n+1} = 2\left(p_n + \frac{1}{2}q_n\right)\left(r_n + \frac{1}{2}q_n\right) \tag{4.4.28}$$

$$r_{n+1} = \left(r_n + \frac{1}{2}q_n\right)^2 \tag{4.4.29}$$

$$\textit{so that } p_{n+1} + q_{n+1} + r_{n+1} = \left(p_n + \frac{1}{2}q_n + \frac{1}{2}q_n + r_n\right)^2 = 1, \tag{4.4.30}$$

as expected. Similarly

$$
\begin{aligned}
p_{n+2} &= \left(p_{n+1} + \frac{1}{2}q_{n+1}\right)^2 \\
&= \left(\left(p_n + \frac{1}{2}q_n\right)^2 + \left(p_n + \frac{1}{2}q_n\right)\left(r_n + \frac{1}{2}q_n\right)\right)^2 \\
&= \left(p_n + \frac{1}{2}q_n\right)^2 \left(p_n + \frac{1}{2}q_n + \frac{1}{2}q_n + r_n\right)^2 \\
&= \left(p_n + \frac{1}{2}q_n\right)^2 = p_{n+1} \qquad (4.4.31)
\end{aligned}
$$

and

$$q_{n+2} = q_{n+1}, \quad r_{n+2} = r_{n+1}$$

so that the proportions of dominants, hybrids and recessives in the $(n+2)$ th generation are same as in the $(n+1)$ th generation.

Thus in any population in which random mating takes place with respect to a characteristic, the proportions of dominants, hybrids and recessive do not change after the first generation. This is known as Hardy-Weinberg law after the mathematician Hardy and geneticist Weinberg who jointly discovered it.

The equations (4.4.27)-(4.4.30) is a set of difference equations of the first order.

### (b) Improvement of Plants through Elimination of Recessives

Suppose the recessives are undesirable and as such we do not allow the recessives in any generation to breed.

Let $p_n, q_n, r_n$ be the proportions of dominants, hybrids and recessives before elimination of recessives and let $p'_n, q'_n, 0$ be the populations after the elimination, then

$$\frac{p'_n}{p_n} = \frac{q'_n}{q_n} = \frac{p'_n + q'_n}{p_n + q_n} = \frac{1}{1 - r_n}$$

Now we allow random mating and let $p_{n+1}, r_{n+1}$ be the proportions in the next generation before elimination of recessives, then using (4.4.27)-(4.4.31)

$$
\begin{aligned}
p_{n+1} &= \left(p'_n + \frac{1}{2}q'_n\right)^2 \\
q_{n+1} &= 2\left(p'_n + \frac{1}{2}q'_n\right)\left(\frac{1}{2}q'_n\right) = q'_n\left(p'_n + \frac{1}{2}q'_n\right) \\
r_{n+1} &= \left(\frac{1}{2}q'_n\right)^2 = \frac{1}{4}q'^2_n
\end{aligned}
$$

After elimination of recessives, let the new proportions be $p'_{n+1}, q'_{n+1}$, so that

$$
\frac{p'_{n+1}}{p_{n+1}} = \frac{q'_{n+1}}{q_{n+1}} = \frac{1}{p_{n+1} + q_{n+1}} = \frac{1}{1 - \frac{1}{4}q'^2_n}
$$

so that

$$
\begin{aligned}
q'_{n+1} = \frac{q'_n\left(p'_n + \frac{1}{2}q'_n\right)}{1 - \frac{1}{4}q'^2_n} &= \frac{q'_n\left(1 - \frac{1}{2}q'_n\right)}{1 - \frac{1}{4}q'^2_n} \\
&= \frac{q'_n}{1 + \frac{1}{2}q'_n}
\end{aligned}
$$

This is a non-linear difference equation of the first order. To solve it we substitute

$$
\begin{aligned}
q'_n &= 1/u_n \\
\text{to get } u_{n+1} &= u_n + \frac{1}{2} \\
\text{which has the solution } u_n &= A + \frac{1}{2}n \\
\text{or } q'_n &= \frac{1}{A + \frac{1}{2}n} \qquad (4.4.32)
\end{aligned}
$$

so that $q'_n \to 0$ and $p'_n \to 1$ as $n \to \infty$. Thus ultimately we should be left with all dominants. Equation (4.4.32) determines the rate at which hybrids disappear. □

# Let us sum up:

- Equations with non-linear differences population growth model equations for non-linear differences.

- Age-structured population models.
- Genetics: Mathematical modelling using difference equations.

## Check your progress:

1. What are types of fixed points ?
2. What are the laws used in the mathematical modelling through difference equations in Genetics?

# 4.5 Difference Equations in Probability Theory

### 4.5.1 Markov Chains

Let a system be capable of being in $n$ possible states $1, 2, \ldots, n$ and let the probability of transition from state $i$ to state $j$ in time interval $t$ to $t+1$ be $p_{ij}$. Let $p_j(t)$ denote the probability that the system is in state $j$ at time $t$ $(j = 1, 2, \ldots, n)$, then at time $t+1$ it can be in any one of the states $1, 2, \ldots, n$.

It can be in the $i$ th state at time $t+1$ in $n$ exclusive ways since it could have been in any one of the $n$ states $1, 2, \ldots, n$ at time $t$ and it could have transited from that state to $i$ th state in time interval $(t, t+1)$. By using the theorems of total and compound probability, we get

$$p_i(t+1) = \sum_{j=1}^{n} p_{ji} p_j(t), \quad i = 1, 2, \ldots, n$$

Or

$$p_1(t+1) = p_{11}p_1(t) + p_{21}p_2(t) + \ldots + p_{n1}p_n(t)$$
$$p_2(t+1) = p_{12}p_1(t) + p_{22}p_2{'}t) + \ldots + p_{n2}p_n(t)$$

$$p_n(t+1) = p_{1n}p_1(t) + p_{2n}p_2(t) + \ldots + p_{nn}p_n(t)$$

or

$$\begin{bmatrix} p_1(t+1) \\ p_2(t+1) \\ \vdots \\ p_n(t+1) \end{bmatrix} = \begin{bmatrix} p_{11} & p_{21} & \dots & p_{n1} \\ p_{12} & p_{22} & \dots & p_{n2} \\ \dots & \dots & \dots & \dots \\ p_{1n} & p_{2n} & \dots & p_{nn} \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ \vdots \\ p_n(t) \end{bmatrix}$$

or

$$P(t+1) = AP(t) \tag{4.5.33}$$

where $P(t)$ is a probability vector and $A$ is a matrix, all of whose elements lie between zero and unity (since these are all probabilities). Further the sum of elements of every column is unity, since the sum of elements of the $i$ th column is $\sum_{j=1}^{n} \cdot p_{ij}$ as this denotes the sum of the probabilities of the system going from the $i$ th state to any other state and this sum must be unity.

The solution of the matrix difference equation (4.5.33) is

$$P(t) = A^t P(0)$$

If all the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $A$ are distinct, we can write

$$A = S\Lambda S^{-1}$$

so that

$$A^t = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

where

$$= S\Lambda^t S^{-1}\Big) \Big(S\Lambda S^{-1}\Big) \dots \Big(S\Lambda S^{-1}\Big)$$

$$= S \begin{bmatrix} \lambda_1^t & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^t & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_n^t \end{bmatrix} S^{-1}$$

The probability vector will not change if $P(t+1) = P(t)$ so that from (4.5.33)

$$(I - A)P(t) = 0$$

Thus if $P$ is the eigenvector of the matrix $A$ corresponding to unit eigenvalue, then $P$

does not change i.e. if the system start with probability vector $P$ at time 0 , it will always remain in this state. Even if the system starts from any other probability vector, it will ultimately be described by the probability vector $P$ as $t \to \infty$.

As a special case, suppose we have a machine which can be in two states, working or non-working. Let the probability of its transition from working to non-working be $\alpha$, of its transition from non-working to working be $\beta$, then the transition probability matrix $A$ is obtained from

$$\begin{array}{c} \\ working \\ nonworking \end{array} \begin{array}{c} \text{working nonworking} \\ \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix} \end{array}$$

The system of difference equations is

$$p_1(t+1) = p_1(t)(1-\alpha) + p_2(t)\beta$$

$$p_2(t+1) = p_1(t)\alpha + p_2(t)(1-\beta)$$

$$\begin{bmatrix} p_1(t+1) \\ p_2(t+1) \end{bmatrix} = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$$

The eigenvalues of the matrix $A$ is given by

$$\begin{vmatrix} 1-\alpha-\lambda & \beta \\ \alpha & 1-\beta-\lambda \end{vmatrix} = 0 \text{ or } (\lambda-1)(\lambda-\overline{1-\alpha-\beta}) = 0$$

The eigenvector corresponding to the unit eigenvalue is $\beta/(\alpha+\beta), \alpha/(\alpha+\beta)$ and as such ultimately the probability of the machines being found in working order is $\beta/(\alpha+\beta)$ and the probability of its being found in a nonworking state is $\alpha/(\alpha+\beta)$. □

### 4.5.2 Gambler's Ruin Problems

Let a gambler with capital $n$ dollars play against an infinitely rich adversary. Let the probability of his winning and losing a unit dollar in any game be $p$ and $q$ respectively where $p+q=1$ and let $p_n$ be the probability of his being ultimately ruined.

At the next game, the probability of his winning is $p$ and if he wins, his capital would become $n+1$ and the probability of his ultimate ruin would be $p_{n+1}$. On the other hand if he loses at the next game, the probability for which is $q$, his capital would become $n-1$ and the probability of his ultimate ruin would be $p_{n-1}$, so that we get the linear difference equation of the second order

$$p_n = pp_{n+1} + qp_{n-1} \tag{4.5.34}$$

The auxiliary equation for this is

$$p\lambda^2 - \lambda + (1-p) = 0$$
$$or\ p(\lambda - 1)\left(\lambda - \frac{1-p}{p}\right) = 0$$

As such the solution of (4.5.34) is

$$p_n = A + B\left(\frac{q}{p}\right)^n \tag{4.5.35}$$

Now let the gambler decide to stop this game when his capital becomes a dollars so that the probability of his being ruined when his starting capital is a dollars is zero i.e. $p_a = 0$. In the same way when his starting capital is zero, he is already ruined, so we put $p_0 = 1$. Using

$$p_0 = 1, \quad p_a = 0$$

(4.5.35) gives

$$p_n = \frac{(q/p)^a - (q/p)^n}{(q/p)^a - 1}$$

Now let $D_n$ denote the expected number of games before the gambler is ruined. If he wins at the next game, his capital becomes $n+1$ and the expected number of games would then be $D_{n+1}$ and if he loses, his capital becomes $n-1$ and the expected number of games would be only $D_{n-1}$. As such, we get

$$D_n = pD_{n+1} + qD_{n-1} + 1 \tag{137}$$

with boundary conditions

$$D_0 = 0, \quad D_a = 0 \tag{138}$$

This gives the solution

$$D_n = \frac{n}{q-p} - \frac{a}{q-p}\frac{1-(q/p)^n}{1-(q/p)^a}$$

□

## Let us sum up:

- Markov chains.
- Gamblers ruin problems.

## Check your progress:

1. What are the two states of machine in Markovs chain Problem?
2. What is the solution of the Gambler's Ruin Problems?

## Summary:

In this unit, we introduced to simple models through difference equations. Also, studied the basic theory of linear difference equations with constant coefficients. In addition, we modelled in economics and finance-population dynamics and genetics. Finally, we solved simple problems.

## Glossary:

Linear difference equation,Complementary function, Particular solution, Economics, Finances, Harrod model, Cobweb model.

## Self Assessment Questions

1. Solve and discuss the behavior of each solution as $t \to \infty$ , $x_{t+2} - 7x_{t+1} + 12x_t = 0$
2. Discuss the stability of the system $x_{t+3} + 9x_{t+2} - 5x_{t+1} + 2x_t = 0$
3. What is the result of Samuelson's Interaction Models?
4. Find the four stable eight-period fixed points.
5. Find the condition for the existence of a three-period fixed point.
6. In a game of chance, the probability of a person winning a second game after losing the first game is a and the probability of his losing a second game after winning the first game is p. Find the ultimate chance of winning.

## Exercises

1. Prove that if the sum of the elements of each column of a square matrix with non-negative elements is less than unity, then all the characteristic roots of this matrix have magnitude less than unity.
2. Write explicitly the conditions that all roots of $a_0\lambda^2 + a_1\lambda + a_2 = 0$ are less than unity in magnitude.
3. Show that the necessary and sufficient conditions for both roots of $m^2+a_1m+a_2 = 0$ to be less than unity in absolute magnitude are $1 + a_1 + a_2 > 0, 1 - a_1 + a_2 > 0, 1 - a_2 > 0$
4. Find the characteristic equation for the Leslie matrix and show that it always has a positive real root. Find the condition that this root is less than unity.

5. Show that if $p = q = 1/2$, the solution of Gamblers Ruin problem is

$$p_n = 1 - n/a$$

# Answers for check your progress

## Section 4.1

1. Population Growth Model, Logistic Growth Model,Prey-Predator Model,Competition Model,Simple Epidemics Model.

## Section 4.2

1. Finding the Linear difference equations by solving the complete function, the particular solution,Obtaining Complementary Function by Use of Matrices, Solution of a System of Linear Homogeneous Difference Equations with Constant Coefficients, Solution of Linear Difference Equations by Using Laplace Transform, Solution of Linear Difference Equations by Using z-Transform, etc.,

## Section 4.3

1. The models proposed are The Harrod Model,The Cobweb Model,Samuelson's Interaction Models .

2. Savings made by the people in a country depend on the national income,The investment depends on the difference between the income of the current year and the last year, All the savings made are invested.

3. Amount of the commodity produced this year and available for sale is a linear function of the price of the commodity in the last year,The price of the commodity this year is a linear function of the amount available this year.

## Section 4.4

1. One period fixed point, two period fixed point, $2^n$ Period fixed point, Fixed point over its period.
2. Hardy- Weinberg Law, Improvement of Plants through Elimination of Recessives.

## Section 4.5

1. Working and non working
2. probability of his ultimate ruin would be $p_{n-1}$, so that we get the linear difference equation of the second order

$$p_n = pp_{n+1} + qp_{n-1}$$

The auxiliary equation for this is

$$p\lambda^2 - \lambda + (1 - p) = 0$$

or

$$p(\lambda - 1)\left(\lambda - \frac{1-p}{p}\right) = 0$$

As such the solution is

$$p_n = A + B\left(\frac{q}{p}\right)^n$$

# References:


1. J.N. Kapur, Mathematical Modelling, Wiley Eastern Limited, New Delhi, 4th Reprint, May 1994.


# Suggested Reading:


1. M. Braun, C.S. Coleman and D. A. Drew, Differential Equation Models, 1994.

2. A.C. Fowler, Mathematical Models in Applied Sciences, Cambridge University Press, 1997.

3. Walter J. Meyer, Concepts of Mathematical Modeling. Courier Corporation, 2012.

4. Edward A. Bender, Introduction to Mathematical Modelling, Dover Publications, 1st ed., 2000.

# UNIT - 5

# Unit 5

# Mathematical Modelling Through Graphs

## Objectives:

- Introduce to simple models through graphs.
- To discuss the mathematical modelling in terms of directed graphs, signed graphs, and weighted digraphs.
- To solve simple problems in graphs.

## 5.1 Mathematical Modelling through graphs

### 5.1.1 Qualitative Relations in Applied Mathematics

It has been stated that ”Applied Mathematics is nothing but solution of differential equations”. This statement is wrong on many counts:

(i) Applied Mathematics also deals with solutions of difference, differential-difference, integral, integro-differential, functional and algebraic equations.

(ii) Applied Mathematics is equally concerned with inequations of all types.

(iii) Applied Mathematics is also concerned with mathematical modelling; in fact mathematical modelling has to precede solution of equations.

(iv) Applied Mathematics also deals with situations which cannot be modelled in terms of equations or inequations; one such set of situations is concerned with qualitative relations.

Mathematics deals with both quantitative and qualitative relationships. Typical qualitative relations are:

$y$ likes $x, y$ hates $x, y$ is superior to $x, y$ is subordinate to $x, y$ belongs to same political party as $x$, set $y$ has a non-null intersection with set $x$; point $y$ is joined to point $x$ by a road, state $y$ can be tansformed into state $x$, team $y$ has defeated team $x, y$ is father of $x$, course $y$ is a prerequisite for course $x$, operation $y$ has to be done before operation $x$, species $y$ eats species $x, y$ and $x$ are connected by an airline, $y$ has a healthy influence on $x$, any increase of $y$ leads to a decrease in $x, y$ belongs to same caste as $x, y$ and $x$ have different nationalities and so on.

Such relationships are very conveniently represented by graphs where a graph consists of a set of vertices and edges joining some or all pairs of these vertices. □

### 5.1.2 The Seven Bridges Problem

There are four land masses $A, B, C, D$ which are connected by seven bridges numbered 1 to 7 across a river (Figure 5.1). The problem is to start from any point in one of the land masses, cover each of the seven bridges once and once only and return to the starting point.

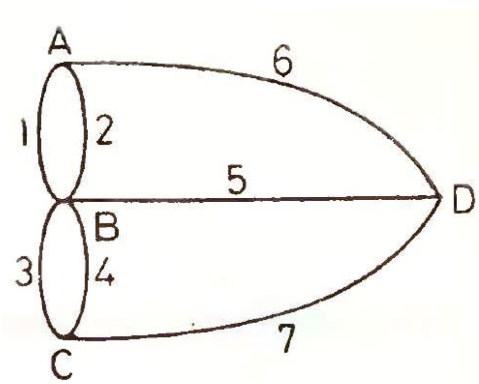


Figure 5.1

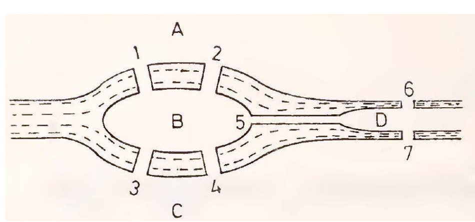


Figure 5.2

There are two ways of attacking this problem. One method is to try to solve the problem by walking over the bridges. Hundreds of people tried to do so in their evening walks and failed to find a path satisfying the conditions of the problem.

A second method is to draw a scale map of the bridges on paper and try to find a path by using a pencil.

It is at this stage that concepts of mathematical modelling are useful. It is obvious that the sizes of the land masses are unimportant, the lengths of the bridges or even whether these are straight or curved are irrelevant.

What is relevant information is that $A$ and $B$ are connected by two bridges 1 and 2, $B$ and $C$ are connected by two bridges 3 and $4, B$ and $D$ are connected by one bridge number 5, $A$ and $D$ are connected by bridge number 6 and $C$ and $D$ are connected by bridge number 7.

All these facts are represented by the graph with four vertices and seven edges in Figure 5.2. If we can trace this graph in such a way that we start with any vertex and return to the same vertex and trace every edge once and once only without lifting the pencil from the paper, the problem can be solved.

Again trial and method cannot be satisfactorily used to show that no solution is possible.

The number of edges meeting at a vertex is called the degree of that vertex. We note that the degrees of $A, B, C, D$ are $3, 5, 3, 3$ respectively and each of these is an odd number.

If we have to start from a vertex and return to it, we need an even number of edges at that vertex. Thus it is easily seen that Konigsberg bridges problem cannot be solved.

This example also illustrates the power of mathematical modelling. □

### 5.1.3 Some Types of Graphs

A graph is called complete if every pair of its vertices is joined by an edge (Figure 5.3).

A graph is called a directed graph or a digraph if every edge is directed with an arrow. The edge joining $A$ and $B$ may be directed from $A$ to $B$ or from $B$ to $A$. If an edge is left undirected in a digraph, it will be assumed to be directed both ways (Figure 5.4).

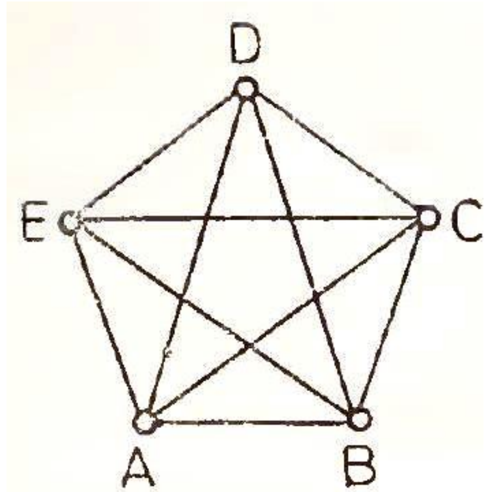


Figure 5.3

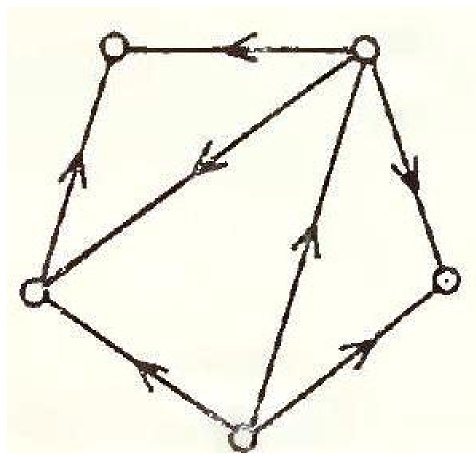

Figure 5.4

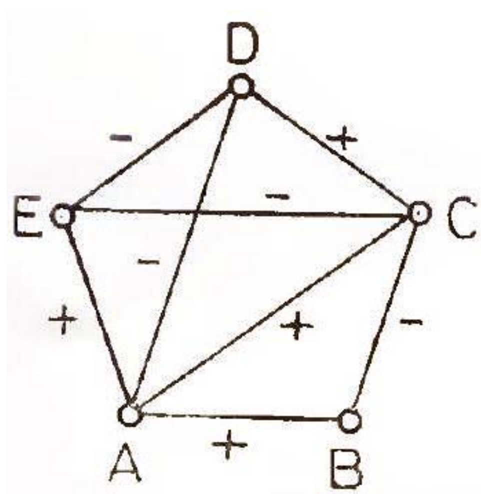


Figure 5.5

A graph is called a signed graph if every edge has either a plus or minus sign associated with it (Figure 5.5).

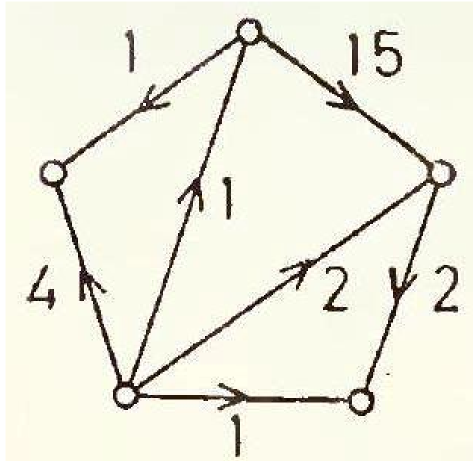


Figure 5.6

A digraph is called a weighted digraph if every directed edge has a weight (giving the importance of the edge) associated with it (Figure 5.6). We may also have digraphs with positive and negative numbers associated with edges. These will be called weighted signed digraphs. □

### 5.1.4 Nature of Models in Terms of Graphs

In all the applications we shall consider, the length of the edge joining two vertices will not be relevant. It will not also be relevant whether the edge is straight or curved. The relevant facts would be

(a) which edges are joined;

(b) which edges are directed and in which direction(s);

(c) which edges have positive or negative signs associated with them;

(d) which edges have weights associated with them and what these weights are. □

# Let us sum up:

- Qualitative relations in applied mathematics.
- The seven bridges problem.
- Character of models in relation to graphs.

## Check your progress:

1. Is it possible or impossible to solve the seven bridges of konigsberg problem?
2. What are the types of graphs we studied in this section?
3. What is the nature of the models in terms of Graphs?

# 5.2 Mathematical Models in Directed Graphs

### 5.2.1 Representing Results of Tournaments

The graph (Figure 5.7) shows that

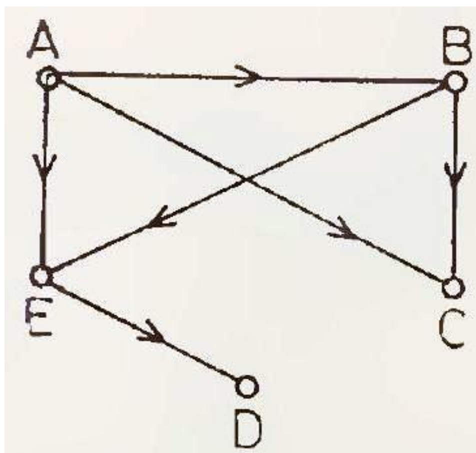


Figure 5.7

(i) Team $A$ has defeated teams $B$, C, $E$.

(ii) Team $B$ has defeated teams $C, E$.

(iii) Team $E$ has defeated $D$.

(iv) Matches between $A$ and $D, B$ and $D, C$ and $D$ and $C$ and $E$ have yet to be played. □

### 5.2.2 One-Way Traffic Problems

The road map of a city can be represented by a directed graph. If only oneway traffic is allowed from point $a$ to point $b$, we draw an edge directed from $a$ to $b$.

If traffic is allowed both ways, we can either draw two edges, one directed from $a$ to $b$ and the other directed from $b$ to $a$ or simply draw an undirected edge between $a$ and $b$.

The problem is to find whether we can introduce one-way traffic on some or all of the roads without preventing persons from going from any point of the city to any other point.

In other words, we have to find when the edges of a graph can be given direction in such a way that there is a directed path from any vertex to every other.

It is easily seen that one-way traffic on the road $DE$ cannot be introduced without disconnecting the vertices of the graph (Figure 5.8).

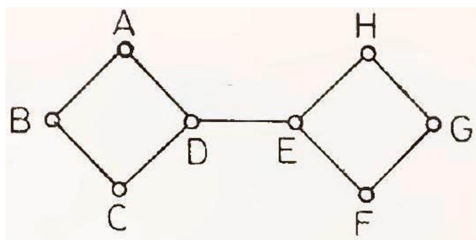


Figure 5.8

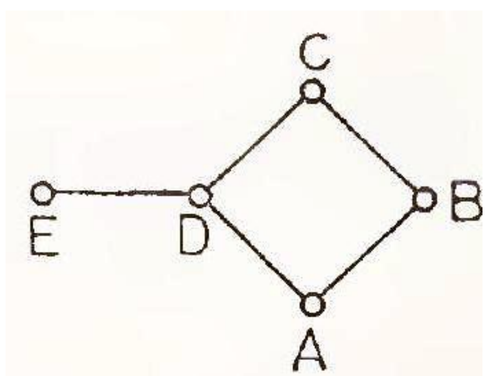


Figure 5.9

In Figure 5.8, $DE$ can be regarded as a bridge connecting two regions of the town. In Figure 5.9 $DE$ can be regarded as a blind street on which a two-way traffic is necessary.

Edges like $DE$ are called separating edges, while other edges are called circuit edges. It is necessary that on separating edges, two-way traffic should be permitted. It can also be shown that this is sufficient. In other words, the following can be established:

If $G$ is an undirected connected graph, then one can always direct the circuit edges of $G$ and leave the separating edges undirected (or both way directed) so that there is a directed path from any given vertex to any other vertex.

### 5.2.3 Genetic Graphs

In a genetic graph, we draw a directed edge from $A$ to $B$ to indicate that $B$ is the child of $A$.

In general each vertex will have two incoming edges, one from the vertex representing the father and the other from the vertex representing the mother.

If the father or mother is unknown, there may be less then two incoming edges. Thus in a genetic graph, the local degree of incoming edges at each vertex must be less than or equal to two.

This is a necessary condition for a directed graph to be a genetic graph, but it is not a sufficient condition.

Thus Figure 5.10 does not give a genetic graph inspite of the fact that the number of incoming edges at each vertex does not exceed two.

Suppose $A_1$ is male, then $A_2$ must be female, since $A_1, A_2$ have a child $B_1$. Then $A_3$ must be male, since $A_2, A_3$ have

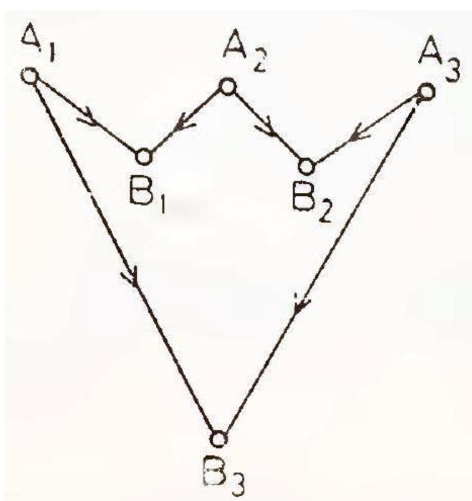


Figure 5.10

Figure 5.10 a child $B_2$. Now $A_1, A_3$ being both males cannot have a child $B_3$.

### 5.2.4 Senior-Subordinate Relationship

If $a$ is senior to $b$, we write $aSb$ and draw a directed edge from $a$ to $b$. Thus the organisational structure of a group may be represented by a graph like the following

[Figure 5.11].

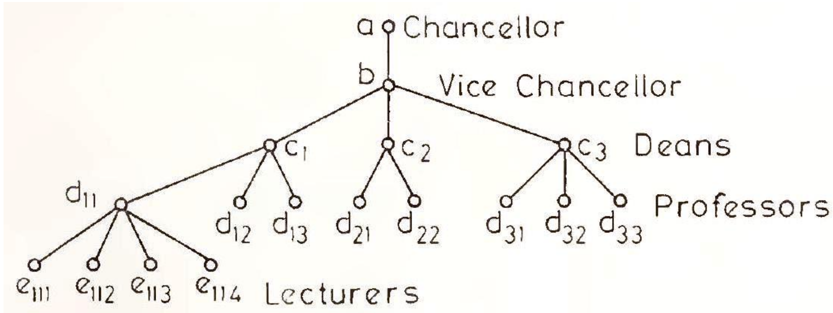


Figure 5.11

The relationship $S$ satisfies the following properties:

(i) $\sim (aSa)$ i.e. no one is his own senior

(ii) $aSb =\sim (bSa)$ i.e. $a$ is senior to $b$ implies that $b$ is not senior to $a$.

(iii) $aSb, bSc \Rightarrow aSc$ i.e. if $a$ is senior to $b$ and $b$ is senior to $c$, then $a$ is senior to $c$.

The following can easily be proved: ”The necessary and sufficient condition that the above three requirements hold is that the graph of an organisation should be free of cycles”.

We want now to develop a measure for the status of each person. The status $m(x)$ of the individual should satisfy the following reasonable requirements.

(i) $m(x)$ is always a whole number

(ii) If $x$ has no subordinate, $m(x) = 0$

(iii) If, without otherwise changing the structure, we add a new individual subordinate to $x$, then $m(x)$ increases

(iv) If, without otherwise changing the structure, we move a subordinate of $a$ to a lower level relative to $x$, then $m(x)$ increases.

A measure satisfying all these criteria was proposed by Harary.

We define the level of seniority of $x$ over $y$ as the length of the shortest path from $x$ to $y$.

To find the measure of status of $x$, we find $n_1$, the number of individuals who are one level below $x, n_2$ the number of individuals who are two levels below $x$ and in general, we find $n_k$ the number of individuals who are $k$ levels below $x$.

Then the Harary measure $h(x)$ is defined by

$$h(x) = \sum_k k n_k \tag{1}$$

It can be shown that among all the measure which satisfy the four requirements given above, Harary measure is the least.

If however, we define the level of senority of $x$ over $y$ as the length of the longest path from $x$ to $y$, and then find $H(x) = \sum_k k n_k$, we get another measure which will be the largest among all measures satisfying the four requirements.

**Problem 5.2.1.** *Determine Harary measure for the following directed graph.*

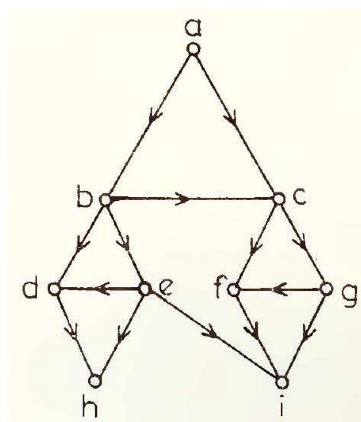


Figure 5.12

**Solution**

For Figure 5.12, we get

| | |
|---|---|
| $h(a) = 1.2 + 4.2 + 2.3 = 16$ | $H(a) = 1.1 + 3.2 + 2.3 + 2.4 = 21$ |
| $h(b) = 1.3 + 2.4 = 11$ | $H(b) = 2.1 + 2.2 + 2.3 + 1.4 = 16$ |
| $h(c) = 1.2 + 1.2 = 4$ | $H(c) = 1.1 + 1.2 + 1.3 = 6$ |
| h(d)=1.1=1 | H(d)=1.1=1 |
| h(e)=1.3=3 | H(e)=1.2+2.1=4 |
| h(f)=1.1=1 | H(f)=1.1=1 |
| h(g)=1.2=2 | H(g)=1.2=2 |
| h(k)=0 | H(k)=0 |
| h(l)=0 | H(l)=0 |

### 5.2.5 Food Webs

**Problem 5.2.2.** *Draw a food web graph.*

**Solution**

Here $aSb$ if $a$ eats $b$ and we draw a directed edge from $a$ to $b$. Here also $\sim(aSa)$ and $aSb \Rightarrow \sim(bSa)$. However the transitive law need not hold. Thus consider the food web in Figure 5.13. Here fox eats bird, bird eats grass, but fox does not eat grass.

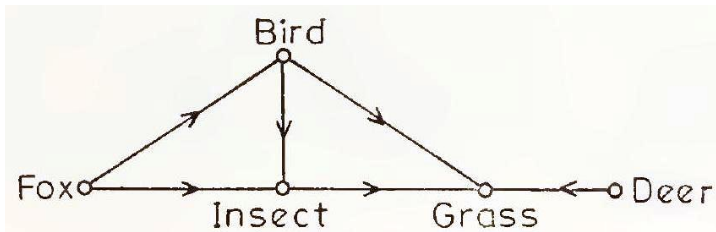


Figure 5.13

We can however calculate measure of the status of each species in this food web by using (1)$h(\text{ bird }) = 2, h(\text{ fox }) = 4, h(\text{ insect }) = 1, h$ (grass ) $= 0$, $h(\text{ deer }) = 1$. □

### 5.2.6 Communication Networks

**Problem 5.2.3.** *How directed graphs can serve as a model for a communication network?*

**Solution**

A directed graph can serve as a model for a communication network. Thus consider the network given in Figure 5.14.

If an edge is directed from $a$ to $b$, it means that $a$ can communicate with $b$.

In the given network $e$ can communicate directly with $b$, but $b$ can communicate with $e$ only indirectly through $c$ and $d$.

However every individual can communicate with every other individual.

Our problem is to determine the importance of each individual in this network. The importance can be measured by the fraction of the messages on an average that pass through him.

In the absence of any other knowledge, we can assume that if an individual can send message direct to $n$ individuals, he will send a message to any one of them with

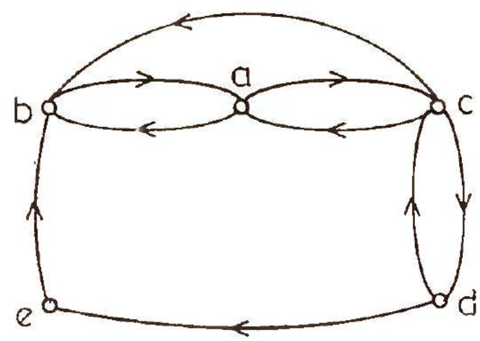


Figure 5.14

probability $1/n$. In the present example, the communication probability matrix is

| | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | 0 | 1/2 | 1/2 | 0 | 0 |
| $b$ | 1/2 | 0 | 1/2 | 0 | 0 |
| $c$ | 1/3 | 1/3 | 0 | 1/3 | 0 |
| $d$ | 0 | 0 | 1/2 | 0 | 1/2 |
| $e$ | 0 | 1 | 0 | 0 | 0 |

No individual is to send a message to himself and so all diagonal elements are zero.

Since all elements of the matrix are non-negative and the sum of elements of every row is unity, the matrix is a stochastic matrix and one of its eigenvalues is unity. The corresponding normalised eigenvector is $[11/45, 13/45, 3/10, 1/10, 1/15]$.

In the long run, these fractions of messages will pass through $a, b, c, d$, $e$ respectively. Thus we can conclude that in this network, $c$ is the most important person.

If in a network, an individual cannot communicate with every other individual either directly or indirectly, the Markov chain is not ergodic and the process of finding the importance of each individual breaks down. □

### 5.2.7 Matrices Associated with a Directed Graph

**Problem 5.2.4.** *How matrices are associated with a directed graph in mathematical modelling?*

**Solution**

For a directed graph with $n$ vertices, we define the $n \times n$ matrix $A = (a_{ij})$ by $a_{ij} = 1$ if there is an edge directed from $i$ to $j$ and $a_{ij} = 0$ if there is no edge directed from $i$ to

$j$.

Thus the matrix associated with the graph of Figure 5.15 is given by

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

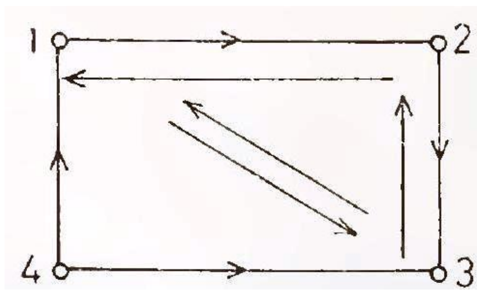


Figure 5.15

We note that

(i) the diagonal elements of the matrix are all zero

(ii) the number of non-zero elements is equal to the number of edges

(iii) the number of non-zero elements in any row is equal to the local outward degree of the vertex corresponding to the row

(iv) the number of nonzero elements in a column is equal to the local inward degree of the vertex corresponding to the column. Now

$$A^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix} \end{array} = (a_{ij}^{(2)})$$

The element $a_{ij}^{(2)}$ gives the number of 2-chains from $i$ to $j$. Thus from vertex 2 to vertex 1 , there are two 2 -chains viz. via vertex 3 and vertex 4 .

We can generalise this result in the form of a theorem viz. "The element $a_{ij}^{(2)}$ of $A^2$ gives the number of 2-chains i.e. the number of paths with two-edges from vertex $i$ to vertex $j$ ".

The theorem can be further generalised to "The element $a_{ij}^{(m)}$ of $A^m$ gives the number of $m$-chains i.e. the number of paths with $m$ edges from vertex $i$ to vertex $j$ ".

It is also easily seen that "The $i$ th diagonal element of $A^2$ gives the number of vertices with which $i$ has symmetric relationship".

From the matrix $A$ of a graph, a symmetric matrix $S$ can be generated by taking the elementwise product of $A$ with its transpose so that in our case

$$S = A \times A^T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

$S$ obviously is the matrix of the graph from which all unreciprocated connections have been eliminated.

In the matrix $S$ (as well as in $S^2, S^3, \dots$ ) the elements in the row and column corresponding to a vertex which has no symmetric relation with any other vertex are all zero. □

### 5.2.8 Application of Directed Graphs to Detection of Cliques

**Problem 5.2.5.** *Explain the application of directed graphs to detection of cliques.*

**Solution**

A subset of persons in a socio-psychological group will be said to form a clique if

(i) every member of this subset has a symmetrical relation with every other member of this subset

(ii) no other group member has a symmetric relation with all the members of the subset (otherwise it will be included in the clique)

(iii) the subset has at least three members.

If other words a clique can be defined as a maximal completely connected subset of the original group, containing at least three persons. This subset should not be properly contained in any larger completely connected subset.

It the group consists of $n$ persons, we can represent the group by $n$ vertices of a graph. The structure is provided by persons knowing or being connected to other persons.

If a person $i$ knows $j$, we can draw a directed edge from $i$ to $j$. If $i$ knows $j$ and $j$ knows $i$, then we have a symmetrical relation between $i$ and $j$.

With this interpretation, the graph of Figure 5.15 shows that persons 1, 2, 3 form a clique. With very small groups, we can find cliques by carefully observing the corresponding graphs.

For larger groups analytical methods based on the following results are useful:

(i) $i$ is a member of a clique if the $i$ th diagonal element of $S^3$ is different from zero.

(ii) If there is only one clique of $k$ members in the group, the corresponding $k$ elements of $S^3$ will be $(k-1)(k-2)/2$ and the rest of the diagonal elements will be zero.

(iii) If there are only two cliques with $k$ and $m$ members respectively and there is no element common to these cliques, then $k$ elements of $S^3$ will be $(k-1)(k-2)/2, m$ elements of $S^3$ will be $(m-1)(m-2)/2$ and the rest of the elements will be zero.

(iv) If there are $m$ disjoint cliques with $k_1, k_2, \ldots, k_m$ members, then the trace of $S^3$ is $\frac{1}{2}\sum_{i=1}^{m} k_i\,(k_i-1)\,(k_i-2)$.

(v) A member is non-cliquical if only if the corresponding row and column of $S^2 \times S$ consists entirely of zeros. □

## Let us sum up:

- One-way traffic problems.
- Genetic graphs.
- Senior-subordinate relationship.
- Food webs.
- Communication networks.
- Matrices connected to a directed graph.
- Utilizing directed graphs for the identification of cliques.

## Check your progress:

1. What are the mathematical models in terms of Directed graphs?

2. How to develop a measure for the status of each person $m(t)$?

# 5.3 Mathematical Models in Signed Graphs

### 5.3.1 Balance of Signed Graphs

A signed (or an algebraic) graph is one in which every edge has a positive or negative sign associated with it. Thus the four graphs of Figure 5.16 are signed graphs. Let positive sign denote friendship and negative sign denote enemity, then in graph (i) $A$ is a friend of both $B$ and $C$ and $B$ and $C$ are

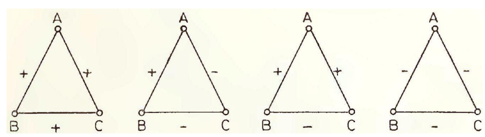

Figure 5.16

also friends. In graph (ii) $A$ is friend of $B$ and $A$ and $B$ are both jointly enemies of $C$. In graph (iii), $A$ is a friend of both $B$ and $C$, but $B$ and $C$ are enemies. In graph (iv) $A$ is an enemy of both $B$ and $C$, but $B$ and $C$ are not friends.

The first two graphs represents normal behaviour and are said to be balanced, while the last two graphs represent unbalanced situations since if $A$ is a friend both $B$ and $C$ and $B$ and $C$ are enemies, this creates a tension in the system and there is a similar tension when $B$ and $C$ have a commion enemy $A$, but are not friends of each other.

We define the sign of a cycle as the product of the signs of component edges. We find that in the two balanced cases, this sign is positive and in the two unbalanced cases, this is negative.

We say that a cycle of length three or a triangle is balanced if and only if its sign is positive.

A complete algebraic graph is defined to be a complete graph such that between any two edges of it, there is a positive or negative sign.

A complete algebraic graph is said to be balanced if all its triangles are balanced. An alternative definition states that a complete algebraic graph is balanced if all its cycles are positive. It can be shown that the two definitions are equivalent.

A graph is locally balanced at a point $a$ if all the cycles passing through $a$ are balanced. If a graph is locally balanced at all points of the graph, it will obviously be balanced.

A graph is defined to be $m$-balanced if all its cycles of length $m$ are positive. For an incomplete graph, it is preferable to define it to be balanced if all its cycles are positive. The definition in terms of triangle is not satisfactory, as there may be no triangles in the graph. □

### 5.3.2 Structure Theorem and Its Implications

**Theorem 5.3.1.** *The following four conditions are equivalent:*

*(i) The graph is balanced i.e. every cycle in it is positive.*

*(ii) All closed line-sequences in the graph are positive i.e. any sequence of edges starting from a given vertex and ending on it and possibly passing through the same vertex more than once is positive.*

*(iii) Any two line-sequences between two vertices have the same sign.*

*(iv) The set of all points of the graph can be partitioned into two disjoint sets such that every positive sign connects two points in the same set and every negative sign connects two points of different sets.*

*Proof.* The last condition has an interesting interpretation with possibility of application.

It states that if in a group of persons there are only two possible relationships viz. liking and disliking and if the algebraic graph representing these relationships is balanced, then the group will break up into two separate parties such that persons within a party like one another, but each person of one party dislikes every person of the other party.

If a balanced situation is regarded as stable, this theorem can be interpreted to imply that a two-party political system is stable. □

### 5.3.3 Antibalance and Duobalance of a Graph

An algebraic graph is said to be antibalanced if every cycles in it has an even number of positive edges. The concept can be obtained from that of a balanced graph by changing the signs of the edges. It will then be seen that an algebraic graph is antibalanced if and only if its vertices can be separated into two disjoint classes, such that each negative edge joins two vertices of the same class and each positive edge joins persons from different classes.

A signed graph is said to be duobalanced if it is both balanced and antibalanced.

### 5.3.4 The Degree of Unbalance of a Graph

**Problem 5.3.2.** *Determine the degrees of unbalance of the following graphs.*

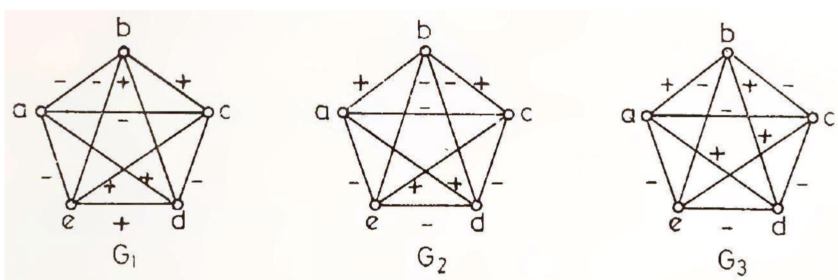


Figure 5.17

**Solution**

For many purposes it is not enough to know that a situation is unbalanced. We may be interested in the degree of unbalance and the possibility of a balancing process which may enable one to pass from an unbalanced to a balanced graph.

The possibility is interesting as it can give an approach to group dynamics and demonstrate that methods of graph theory can be applied to dynamic situations also.

Cartwright and Harary define the degree of balance of a group $G$ to be the ratio of the positive cycles of $G$ to the total number of cycles in $G$.

This balance index obviously lies between 0 and 1. $G_1$ has six negative triangles viz $(abc), (ade), (bcd), (bce), (bde), (cde)$ and has four positive triangles.

$G_2$ has four negative triangles viz $(abc), (abd), (bce)$ and $(bde)$ and six positive triangle. The degree of balance of $G_1$ is therefore less than the degree of balance of $G_2$.

However in order to get a balanced graph from $G_1$, we have to change the sign of only two edges viz. $bc$ and $de$ and similarly to make $G_2$ balanced we have to change the signs of two edges viz $bc$ and $bd$. From this point of view both $G_1$ and $G_2$ are equally unbalanced (Figure 5.17)

Abelson and Rosenberg therefore gave an alternative definition. They defined the degree of unbalance of an algebraic graph as the number of the smallest set of edges of $G$ whose change of sign produces a balanced graph.

The degree of an antibalanced complete algebraic graph (i.e. of a graph all of whose triangles are negative) is given by $[n(n-2)+k]/4$ where $k=1$ if $n$ is odd and $k=0$ if $n$ is even.

It has been conjectured that the degree of unbalancing of every other complete algebraic graph is less than or equal to this value. □

## Let us sum up:

- Balance of signed graphs.
- Structure theorem.
- The Degree of unbalance of a graph.

## Check your progress:

1. Explain the mathematical models in signed graphs.
2. What is the unbalanced definition given by Abelson and Rosenberg?
3. What are the conditions of Structure Theorem?

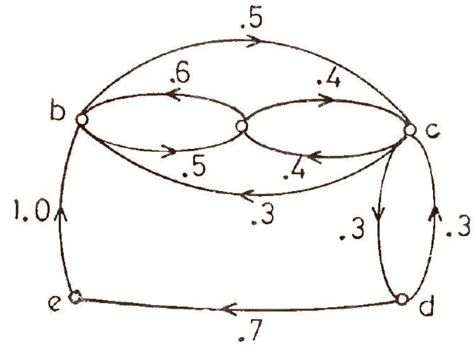


Figure 5.18

## 5.4 Mathematical Modelling in Weighted Digraphs

### 5.4.1 Communication Networks with Known Probabilities of Communication

In the communication graph of Figure 5.18, we know that $a$ can communicate with both $b$ and $c$ only and in the absence of any other knowledge, we assigned equal probabilities to $a$ 's communicating with $b$ or $c$.

However we may have a priori knowledge that $a$ 's chances of communicating with $b$ and $c$ are in the ratio 3 : 2, then we assign probability 6 to $a$'s communicating with $b$ and .4 to $a$ 's communicating with $c$.

Similarly we can associate a probability with every directed edge and we get the weighted digraph (Figure 5.18) with the associated matrix

$$
B = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}
\begin{array}{c}
\begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{bmatrix}
0 & 0.6 & 0.4 & 0 & 0 \\
0.5 & 0 & 0.5 & 0 & 0 \\
0.4 & 0.3 & 0 & 0.3 & 0 \\
0 & 0 & .3 & 0 & 0.7 \\
0 & 1.0 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

We note that the elements are all non-negative and the sum of the elements of every row is unity so that $B$ is a stochastic matrix and unity is one of its eigenvalues.

The eigenvector corresponding to this eigenvalues will be different from the eigenvector and so the relative importance of the individuals depends both on the directed edges as well as on the weights associated with the edges.

### 5.4.2 Weighted Digraphs and Markov Chains

A Markovian system is characterised by a transition probability matrix. Thus if the states of a system are represented by $1, 2, \ldots, n$ and $p_{ij}$ gives the probability of transition from the $i$th state to $j$ th state, the system is characterised by the transition probability matrix (t.p.m)

$$T = \begin{bmatrix} p_{11} & p_{12} & \ldots & p_{1j} & \ldots & p_{1n} \\ p_{21} & p_{22} & \ldots & p_{2j} & \ldots & p_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ p_{i1} & p_{i2} & \ldots & p_{ij} & \ldots & p_{in} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ p_{n1} & p_{n2} & \ldots & p_{nj} & \ldots & p_{nn} \end{bmatrix} \tag{7}$$

Since $\sum_{i=1}^{n} p_{ij}$ represents the probability of the system going from $i$ th state to any other state or of remaining in the same state, this sum must be equal to unity. Thus the sum of elements of every row of a t.p.m. is unity.

Consider a set of $N$ such Markov systems where $N$ is large and suppose at any instant $NP_1, NP_2, \ldots, NP_n$ of these $(P_1 + P_2 + \ldots + P_n = 1)$ are in states $1, 2, 3, \ldots, n$ respectively.

After one step, let the proportions in these states be denoted by $P'_1, P'_2, \ldots, P'_n$, then

$$P'_1 = P_1 p_{11} + P_2 p_{21} + P_3 p_{31} + \ldots\ldots + P_n p_{n1}$$
$$P'_2 = P_2 p_{12} + P_2 p_{22} + P_3 p_{32} + \ldots\ldots + P_n p_{n2}$$
$$P'_n = P_1 p_{1n} + P_{2p_{2n}} + P_{3p_{3n}} + \ldots\ldots + P_n p_{nn}$$

$$\text{or} \qquad P' = PT \tag{9}$$

where $P$ and $P'$ are row matrices representing the proportions of systems in various states before and after the step and $T$ is the t.p.m.

We assume that the system has been in operation for a long time and the proportions $P_1, P_2, \ldots, P_n$ have reached equilibrium values. In this case

$$P = PT \text{ or } P(I - T) = 0, \tag{10}$$

where $I$ is the unit matrix. This represents a system of $n$ equations for determining the equilibrium values of $P_1, P_2, \ldots, P_n$. If the equations are consistent, the determinant of the coefficient must vanish i.e. $|T - I| = 0$. This requires that unity must be an eigenvalue of $T$. However this, as we have seen already is true. This shows that an equilibrium state is always possible for a Markov chain.

A Markovian system can be represented by a weighted directed graph. Thus consider the Markovian system with the stochastic matrix

| | *a* | *b* | *c* | *d* |
|---|---|---|---|---|
| *a* | 0.2 | 0.8 | 0 | 0 |
| *b* | 0.3 | 0.6 | 0.1 | 0 |
| *c* | 0.2 | 0.4 | 0.3 | 0.1 |
| *d* | 0 | 0 | 0 | 1 |

Its weighted digraph is given in Figure 5.19.

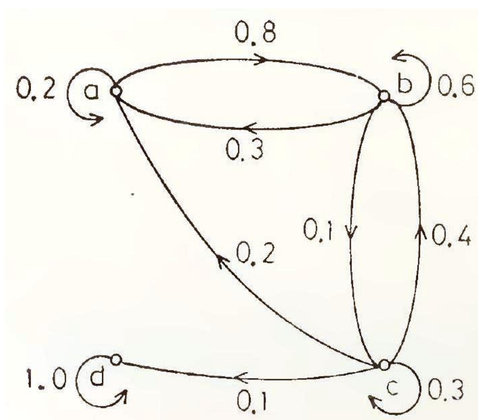


Figure 5.19

In this example $d$ is an absorbing state or a state of equilibrium. Once a system reaches the state $d$, it stays there for ever.

It is clear from Figure 5.19, that in whichever state, the system may start, it will ultimately end in state $d$. However the number of steps that may be required to reach $d$ depends on chance.

Thus starting from $c$, the number of steps to reach $d$ may be $1, 2, 3, 4, \ldots$; starting from $b$ the number of steps to reach $d$ may be $2, 3, 4, \ldots$ and starting for $a$, the number of steps may be $3, 4, 5, \ldots$ In each case, we can find the probability that the number of steps required is $n$ and then we can find the expected number of steps to reach it.

Thus for the matrix

$$\begin{array}{cc} & \begin{array}{cc} a & \quad b \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 0 \\ 1/3 & 2/3 \end{bmatrix} \end{array}$$

$a$ is an absorbing state. Starting from $b$, we can reach $a$ in $1, 2, 3, \ldots, n$ steps with probabilities $(1/3), (1/3)(2/3), (1/3)(2/3)^2, \ldots, (1/3)(2/3)^{n-1}, \ldots$, so that tbe expected number of steps is

$$\sum_{n=1}^{\infty} n \frac{1}{3} \left(\frac{2}{3}\right)^{n-1} = 3 \tag{13}$$

□

### 5.4.3 General Communication Networks

So for we have considered communication networks in which the weight associated with a directed edge represents the probability of communication along that edge. We can however have more general networks e.g.

(a) for communication of messages where the directed edge represents the channel and the weight represents the capacity of the channel say in bits per second

(b) for communication of gas in pipelines where the weights are the capacities, say in gallons per hour

(c) communication roads where the weights are the capacities is cars per hour.

An interesting problem is to find the maximum flow rate, of whatever is being communicated, from any vertex of the communication network to any other. Useful graph-theoretic algorithms for this have been developed by Elias. Feinstein and Shannon as well as by Ford and Fulkerson.

### 5.4.4 More General Weighted Digraphs

In the most general case, the weight associated with a directed edge can be positive or negative. Thus Figure 5.20 means that a unit change at vertex 1 at time $t$ causes changes of -2 units at vertex 2 , of 2 units at vertex 4 and of 3 units at vertex 5 at time $t+1$.

Similarly a change of 1 unit

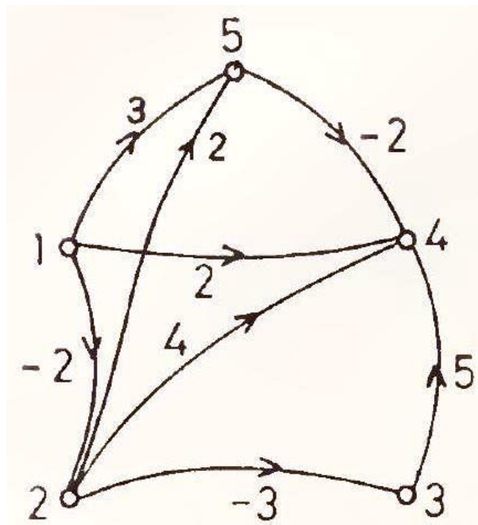


Figure 5.20

Figure 5.20 at vertex 2 causes a change of -3 units at 3 vertex, 4 units at vertex 4 and of 2 units at vertex 5 and so on.

Given the values at all vertices at time $t$, we can find the values at times $t+1, t+2, t+3, \ldots$ The process of doing this systematically is known as the pulse rule.

These general weighted digraphs are useful for representing energy flows, monetary flows and changes in environmental conditions. □

### 5.4.5 Signal Flow Graphs

**Problem 5.4.1.** *Represent the algebraic equations* $x_1 = 4y_0 + 6x_2 - 2x_3$, $x_2 = 2y_0 - 2x_1 + 2x_3$, $x_3 = 2x_1 - 2x_2$ *weighted digraph.*

**Solution**

The system of given algebraic equations

$$\begin{aligned} x_1 &= 4y_0 + 6x_2 - 2x_3 \\ x_2 &= 2y_0 - 2x_1 + 2x_3 \\ x_3 &= 2x_1 - 2x_2 \end{aligned} \tag{14}$$

can be represented by the weighted digraph in Figure 5.21.

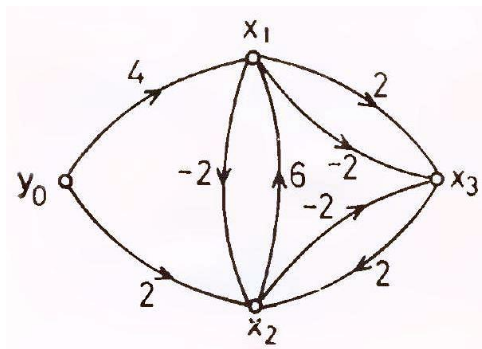


Figure 5.21

For solving for $x_1$, we successively eliminate $x_3$ and $x_2$ to get the graphs in Figure 5.22. Finally we get

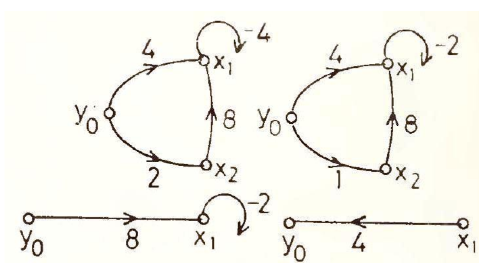


Figure 5.22

$$x_1 = 4y_0$$

We can similarly represent the solution of any number of linear equations graphically.

□

### 5.4.6 Weighted Bipartitic Digraphs and Difference Equations

**Problem 5.4.2.** *Represent the system of difference equations* $x_{t+1} = a_{11}x_t + a_{12}y_t + a_{13}z_t$, $y_{t+1} = a_{21}x_t + a_{22}y_t + a_{23}z_t$, $z_{t+1} = a_{31}x_t + a_{32}y_t + a_{33}z_t$ *by weighted bipartite digraph.*

**Solution**

Consider the system of difference equations

$$\begin{aligned} x_{t+1} &= a_{11}x_t + a_{12}y_t + a_{13}z_t \\ y_{t+1} &= a_{21}x_t + a_{22}y_t + a_{23}z_t \\ z_{t+1} &= a_{31}x_t + a_{32}y_t + a_{33}z_t \end{aligned} \tag{15}$$

This can be represented by a weighted bipartitic digraph (Figure 5.23). The weights can be positive or negative.

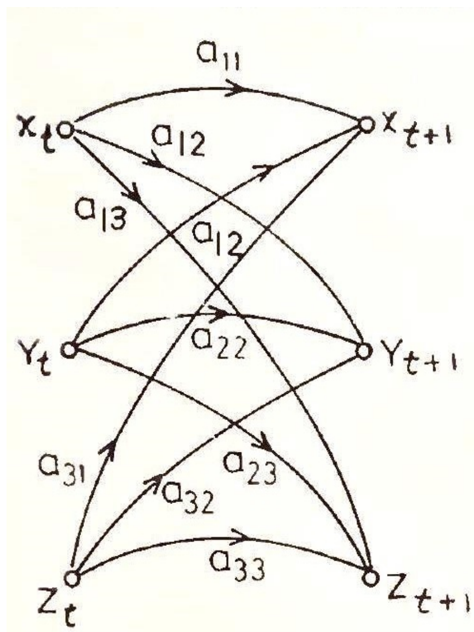


Figure 5.23

□

# Let us sum up:

- Communication networks with predetermined communication probabilities.
- Weighted digraphs and markov chains.
- General communication networks.

- Signal flow graphs.
- Weighted bipartitic digraphs and difference equations.

## Check your progress:

1. What is the expected number of steps in Markovian system?
2. What is the weighs of the Weighted Bipartitic Digraphs?

## Summary:

In this unit, we introduced to simple models through graphs. Also, discussed the mathematical modelling in terms of of directed graphs, signed graphs, and weighted digraphs. Finally, we solved simple problems in graphs.

## Glossary:

Seven bridges problem, One way traffic problem, Genetic graphs, Food webs, Directed graphs, Signed graphs, Weighted digraphs.

## Self Assessment Questions

1. In the Konigsberg problem suggest deletion or addition of minimum number of bridges which may lead to a solution of the problem
2. An intelligence officer can communicate with each of his n subordinates and each subordinate can communicate with him, but the subordinates cannot communicate among themselves. Draw the graph and find the importance of each subordinate relative to the officer.
3. Draw Balanced and unbalanced graph by Structure Theorem.

4. Give the Graphical solution of

$$x_1 - 2x_2 + 3x_3 = 2$$
$$3x_1 + x_2 - x_3 = 3$$
$$x_1 + 2x_2 + x_3 = 4$$

# Exercises

1. A graph is called regular if each of its vertices has same degree r. Draw regular graphs with 6 vertices and degree 5, 4 and 3.

2. Show that in Konigsberg, four one-way bridges will be enough to connect the four land masses.

3. Enumerate all possible four-cliques.

4. Show that a signed graph has an idealised party structure if and only no circuit has exactly one - sign.

5. Show that if all cycles of a signed graph are positive, then all its cycles are also positive. State and prove its converse also.

# Answers for check your progress

## Section 5.1

1. With the original layout of the seven bridges of Konigsberg, it is impossible to find a path that crosses each and every bridge once as both the people of Konigsberg discovered by trial and error and as Euler discovered using proofs based in the branch of mathematics known as graph theory. However, be adding or removing one or more bridges a path can be found and can depending on the number and choice of bridge result in a circuit being possible.

2. Complete graph, Directed graph or Digraph, Signed Graph, Weighted digraph, weight signed digraph.

3. (a) which edges are joined; (b) which edges are directed and in which direction(s); (c) which edges have positive or negative signs associated with them; (d) which edges have weights associated with them and what these weights are.

## Section 5.2

1. Representing Results of Tournaments, One-Way Traffic Problems, Genetic Graphs, Senior-Subordinate Relationship etc.

2. (i) $m(x)$ is always a whole number (ii) If $x$ has no subordinate, $m(x) = 0$ (iii) If, without otherwise changing the structure, we add a new individual subordinate to $x$, then $m(x)$ increases (iv) If, without otherwise changing the structure, we move a subordinate of a to a lower level relative to $x$, then $m(x)$ increases.

## Section 5.3

1. The mathematical models in signed graphs deals with Balance of Signed Graphs, Structure Theorem and Its Implications, Antibalance and Duo balance of a Graph, The Degree of Unbalance of a Graph.

2. The degree of unbalance of an algebraic graph as the number of the smallest set of edges of G whose change of sign produces a balanced graph.

3. The graph is balanced i.e. every cycle in it is positive.(ii) All closed line-sequences in the graph are positive i.e. any sequence of edges starting from a given vertex and ending on it and possibly passing through the same vertex more than once is positive. (iii) Any two line-sequences between two vertices's have the same sign. (iv) The set of all points of the graph can be partitioned into two disjoint sets such that every positive sign connects two points in the same set and every negative sign connects two points of different sets

## Section 5.4

1. $\sum_{n=1}^{\infty} n\frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = 3$

2. The weights can be positive or negative.

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